5. Odd and Even

Definition 5.1. *We say an integer n is **even** if it is divisible by* 2.

Remark 5.2. *Note that definitions are always implicitly if and only if statements, even if this is not explicit.*

Definition 5.3. *We say an integer n is **odd** if it is not even.*

In other words, n is odd if it is not divisible by 2.

Proposition 5.4. *Let n be an integer.*

Then n is odd if and only if there is an integer k such that $n = 2k+1$.

Proof. We first do the direction ($\Longleftarrow$).

Suppose not. We will derive a contradiction. Suppose that $n = 2k+1$ and n is even (that is, not odd). As n is even, there is an integer l such that $n = 2l$. We have

$$2l = n = 2k + 1.$$

It follows that

$$2(l - k) = 1.$$

Thus 2 divides 1. As $2 > 1$ this is a contradiction. It follows that if $n = 2k + 1$ then n is odd.

Now we prove the other direction ($\Longrightarrow$). By assumption n is odd. Let m be the largest even integer less than or equal to n. As m is even, there is an integer k such that $m = 2k$. Let $r = n - m$. Then r is a non-negative integer.

Claim 5.5. $r > 0$.

Proof of (5.5). Suppose not. If $r = 0$ then $n = m$ and so n is even, a contradiction. □

Claim 5.6. $r \leq 1$.

Proof of (5.6). Suppose not. Then $r > 1$. As r is an integer it follows that $r \geq 2$. In this case

$$\begin{aligned} m + 2 &\leq m + r \\ &\leq n. \end{aligned}$$

On the other hand,

$$\begin{aligned} m + 2 &= 2k + 2 \\ &= 2(k + 1). \end{aligned}$$

Thus $m + 2$ is an even integer less than or equal to n. This contradicts our choice of m. Thus $r \leq 1$. □

As r is integer and $0 < r \leq 1$ we must have $r = 1$. But then $n = m + 1 = 2k + 1$. □

Corollary 5.7. *An integer n is odd if and only if $n + 1$ is even.*

Proof. We first prove ($\Longrightarrow$).

If n is odd then (5.4) implies that there is an integer k such that $n = 2k + 1$. In this case

$$\begin{aligned} n + 1 &= 2k + 1 + 1 \\ &= 2k + 2 \\ &= 2(k + 1). \end{aligned}$$

Thus $n + 1$ is even.

Now we prove ($\Longleftarrow$).

If $n + 1$ is even then there is an integer k such that $n + 1 = 2k$. In this case

$$\begin{aligned} n &= 2k - 1 \\ &= 2(k - 1) + 2 - 1 \\ &= 2(k - 1) + 1. \end{aligned}$$

As $k - 1$ is an integer, (5.4) implies that n is odd. □