21. Rational and Irrationals

We have already seen that Cantor proved that there are irrational numbers simply by counting; the rationals are countable and the reals are not.

Theorem 21.1. $\sqrt{2}$ *is irrational.*

Proof. Suppose not. Then we may integers a and b, $b \neq 0$ such that

$$\frac{a}{b} = \sqrt{2}.$$

If a is negative then b is negative as well. If we replace a by $-a$ and b by $-b$ we reduce to the case that both a and b are positive integers.

If we multiply through by b then we get

$$a = b\sqrt{2}.$$

Squaring both sides we get

$$a^2 = 2b^2.$$

As the RHS is even, the LHS is even as well.

Suppose that a is odd. Then there is an integer k such that $a = 2k+1$. In this case

$$\begin{aligned} a^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2l + 1, \end{aligned}$$

where $l = (2k^2 + 2k)$ is an integer. It follows that a^2 is odd. As the RHS is even, this is impossible.

Thus a is even. It follows that there is an integer c such that $a = 2c$. In this case

$$a^2 = 4c^2,$$

and so

$$4c^2 = 2b^2.$$

Cancelling a factor of 2, we get

$$2c^2 = b^2.$$

We started with an expression of the form $a^2 = 2b^2$ and we derived a new relation of the form $2c^2 = b^2$. Note that $b < a$.

Let

$$A = \{\, a \in \mathbb{N} \,|\, \text{we may find a natural number } b \text{ such that } a^2 = 2b^2 \,\}.$$

As A is a non-empty subset of $\mathbb{N}$, it follows that a smallest element a. But we have already proved that given a we may find a smaller element b of A, a contradiction.

Thus $\sqrt{2}$ is irrational. □

Theorem 21.2 (Dirichlet's Theorem). *Let α be an irrational number.*

Then there are infinitely many integers p and q such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

Proof. It suffices to prove that given any positive integer N there are integers p and $1 \leq q \leq N$ such that

$$|q\alpha - p| < \frac{1}{N}.$$

Indeed, if we divide through by q, and use the fact that $q \leq N$ we get

$$\begin{aligned} |\alpha - \frac{p}{q}| &= \frac{1}{q}|q\alpha - p| \\ &< \frac{1}{qN} \\ &\leq \frac{1}{q^2}. \end{aligned}$$

On the other hand, if we have constructed pairs (p_i, q_i), $1 \leq i \leq k$, and we choose M so that

$$|q_i\alpha - p_i| > \frac{1}{M},$$

then any pair we construct so that

$$|q\alpha - p| \leq \frac{1}{M},$$

is different from the first k pairs. Thus we get infinitely many pairs this way.

Consider the $N+1$ numbers $q\alpha$, where $0 \leq q \leq N$. Let $d_0, d_1, d_2, \dots, d_N$ be the decimal part of each of these numbers. Each one of these numbers lies in one of the N intervals

$$[0, \frac{1}{N}), \quad [\frac{1}{N}, \frac{2}{N}), \quad [\frac{2}{N}, \frac{3}{N}), \quad \dots \quad [\frac{N-1}{N}, \frac{1}{N}).$$

Since have $N+1$ pigeons (the decimal parts) and N pigeons (the intervals), two of the decimals must lie in the same interval.

Suppose that the two decimals are d_r and d_s, where $0 \leq s < r \leq N$. It follows that the decimal part of $(r-s)\alpha$ is less than $1/N$, that is

$(r-s)\alpha$ is no further than $1/N$ from an integer. Call the integer p and let $q=(r-s)$. Then $0 \leq q \leq N$ and

$$|q\alpha - p| < \frac{1}{N}. \qquad \square$$