2. Königsberg bridge problem

A mathematician is a machine for turning coffee into theorems

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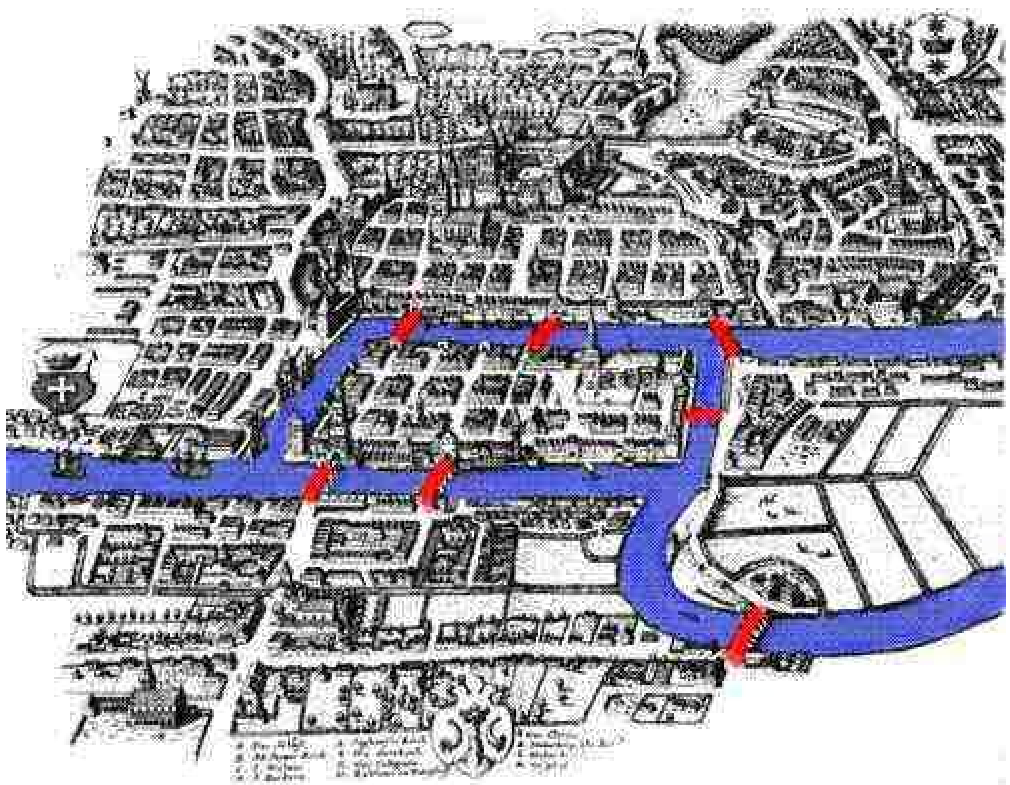
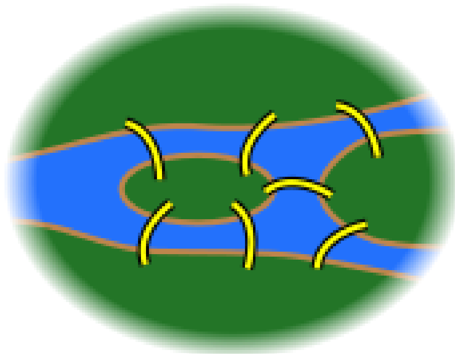
Königsberg was a small town in Prussia. There is a river running through the town and there were seven bridges across the river. The inhabitants of Königsberg liked to walk around the town and cross all of the bridges:



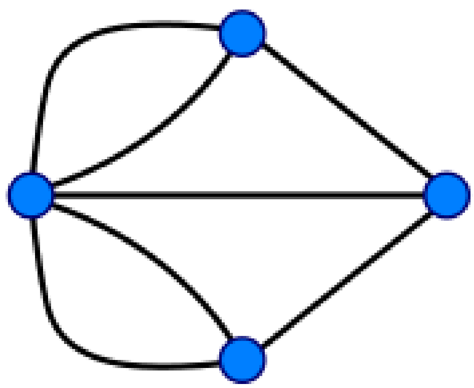
Question 2.1. *Is it possible to walk around the town and cross every bridge, once and once only?*

After quite a bit of trial and error, one can guess that the answer is no. Euler thought about this question and came up with a beautiful solution, with far reaching consequences.

The first step is to transform this problem a little bit, leaving only the essence of the problem.



Now we replace every landmass by a vertex and every bridge by an edge.



The resulting figure is called a **graph**. In fact we can give a general definition of a graph:

Definition 2.2. *A graph G consists of a pair (V, E) of sets. The elements of V are called the* ***vertices****. The elements of E are called the* ***edges****. Every edge is an unordered pair of vertices, xy. x and y are called the* ***endpoints*** *of the edge xy.*

If we call the vertices a, b, c and d, starting at the top and going clockwise, the graph above is specified by

$$V = \{ a, b, c, d \} \qquad \text{and} \qquad E = \{ ab, ad, (ad)', bc, bd, cd, (cd)' \}.$$

Definition 2.3. *Let G be a graph and let v be a vertex of G.*

The ***degree of*** *v, denoted $d(v)$, is the number of edges with v as an endpoint.*

In the graph above, we have

$$d(a) = 3, \qquad d(b) = 3, \qquad d(c) = 3, \qquad \text{and} \qquad d(d) = 5.$$

Definition 2.4. *Let G be a graph. A* ***walk*** *is a sequence*

$$v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n,$$

where $v_1, v_2, \dots, v_n$ are vertices of G and $e_1, e_2, \dots, e_{n-1}$ are edges of G such that $e_i = v_i v_{i+1}$. A ***trail*** *is a walk that does not repeat any edges. An* ***Euler trail*** *is a trail that uses every edge.*

A ***ciruit*** *is a walk that begins and ends at the same vertex.*

We say a graph G is ***connected*** *if there is a walk between any pair of vertices.*

Theorem 2.5 (Euler)**.** *Let G be a connected graph.*

Then

(1) there is an Euler trail (that is, an Euler trail, which is also a circuit) of G if and only if every vertex of G has even degree.
(2) there is an Euler trail of G which starts at the vertex w and ends at a different vertex $v \neq w$ a different vertex if and only if these are the only vertices of odd degree.

Proof. We first prove (1). We only prove that the condition on the degrees is necessary, that is, we prove that if there is an Euler trail that starts and end at the same place then every vertex has even degree.

Let

$$v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n,$$

be an Euler circuit, so that $v_1 = v_n$.

Pick a vertex v. Call an edge incoming (or label it blue) if we go *from* another vertex *to* v along the walk. Call an edge outgoing (or label it red) if we go *from* v *to* another vertex. Note that since we use every edge, every edge with endpoint v receives a colour.

On the other hand, note that we visit the vertex v as many times as we leave it. For any vertex other than $v = v_1 = v_n$, this is clear, since we leave the vertex immediately after we get there. The vertex v_1 is special. At the very first step we leave the vertex but then again at the very last step we return to v_1.

So every edge receives the same number of blue and red edges. Therefore the degree of every vertex is even. Thus we have proved the $\implies$ direction of (1).

We can prove that the degree condition of (2) is necessary, along the same lines as (1).

However, here is a much sneakier way to prove the $\implies$ direction of (2). Define a new graph G' by adding another edge, an edge between v and w. G' is connected as G is connected. The degree of every vertex of G' is unchanged except for the degrees of v and w and for these vertices we increased the degree by one. Since these were the only vertices of odd degree, every vertex in G' has even degree and so there is an Euler circuit in G',

$$v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n.$$

This Euler circuit uses the edge $e = wv$, since it uses every edge in G'. We can start and end a circuit at any point and with any edge, so we may assume that $v_{n-1} = v$, $e_{n-1} = vw$ and $v_n = w$.

Then

$$v_1, e_1, v_2, e_2, \ldots, v_{n-2}, e_{n-2}, v_{n-1},$$

is an Euler trail in G that starts with w and ends with v. $\square$

In the example above, there are four vertices and every vertex has odd degree. So it is not possible to walk around Königsberg and use every bridge once and once only.