18. Countable and uncountable

The first really interesting result about two sets having the same cardinality is the following:

Proposition 18.1. *$\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality.*

Proof. We have to define a bijection

$$f \colon \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}.$$

The easiest way to prove this is by a picture. □

Since proofs by pictures are not really supposed to be allowed, we need another way to prove (18.1). The following is very useful:

Theorem 18.2 (Schröder-Bernstein)**.** *If A and B are two sets and $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow A$ are two injective functions then A and B have the same cardinality.*

Proof of (18.1). **Aliter** We just find injective functions both ways. The function

$$f \colon \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N} \qquad \text{given by} \qquad f(n) = (n, 0),$$

is clearly injective. Consider the function

$$g \colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \qquad \text{given by} \qquad g(a, b) = 2^a 3^b.$$

Suppose that $g(a, b) = g(c, d)$. Then $2^a 3^b = 2^c 3^d$. Comparing powers of two and powers of three, we must have $a = c$ and $b = d$ (see (18.3)). Thus g is injective, since factorisation of natural numbers into primes is injective. Now apply (18.2). □

Theorem 18.3 (Fundamental Theorem of Arithmetic)**.** *Every integer $n \in \mathbb{Z}$ has a factorisation into a product of ± 1 and primes,*

$$n = \pm p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}.$$

This factorisation is unique, if we order the primes in ascending order.

Theorem 18.4. *The set of rational numbers $\mathbb{Q}$ is countable.*

Proof. The natural inclusion

$$i \colon \mathbb{Z} \longrightarrow \mathbb{Q} \qquad \text{given by} \qquad i(n) = n,$$

is an injective map.

Define a map

$$f \colon \mathbb{Q} \longrightarrow \mathbb{Z} \times \mathbb{Z},$$

by sending the rational number q to the pair (a, b), where $b > 0$ and $q = a/b$ and a and b are coprime, that is, they have no common prime factors. Suppose that $f(q_1) = f(q_2)$. If $f(q_i) = (a_i, b_i)$, $i = 1, 2$, then $(a_1, b_1) = (a_2, b_2)$ so that

$$\begin{aligned} q_1 &= \frac{a_1}{b_1} \\ &= \frac{a_2}{b_2} \\ &= q_2. \end{aligned}$$

Thus f is injective. We have already seen that $\mathbb{Z}\times\mathbb{Z}$ is countable and so there is an injective map $\mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}$. The composition $g \circ f \colon \mathbb{Q} \longrightarrow \mathbb{Z}$ is an injective map, as the composition of injective maps is injective.

Now apply (18.2) to conclude that $\mathbb{Q}$ is countable. □

Theorem 18.5 (Cantor's diagonalisation argument)**.** *The real numbers are uncountable.*

Uncountable means not countable.

We will need a basic result about how to represent real numbers, which we will accept without proof:

Lemma 18.6. *Every real number r has a decimal expansion, so that we can write*

$$r = n + 0.d_1 d_2 \dots,$$

where $n \in \mathbb{Z}$ is an integer and the digits $d_1, d_2, \dots$ are integers between zero and 9. Moreover, given any k, we may find a digit $d_l \neq 9$, $l > k$.

The last statement just means we can always replace repeating 9's.

$$0.99999\cdots = 1,$$

and so on.

Proof of (18.5). Suppose not, suppose that the real numbers are countable. Then there would be a surjective function $f\colon \mathbb{N} \longrightarrow \mathbb{R}$. Then we get a list of all real numbers, $r_0, r_1, r_2, \dots$, where $r_i = f(i)$. Imagine making an actual list of these numbers

$$\begin{aligned} r_0 &= m_0 + 0.a_{01}a_{02}a_{03}\dots \\ r_1 &= m_1 + 0.a_{11}a_{12}a_{13}\dots \\ r_2 &= m_2 + 0.a_{21}a_{22}a_{23}\dots \\ \vdots &= \vdots \\ r_n &= m_n + 0.a_{n1}a_{n2}a_{n3}\dots. \end{aligned}$$

We construct another real number r,

$$r = m + 0.a_1 a_2 a_3 \dots,$$

as follows.

If the integer part m_0 of r_0 is not zero, then we let the integer part m of r be zero. If the integer part m_0 of r_0 is zero then we let the integer part m of r be one. Thus $r \in [0, 2]$. The digits $a_1, a_2, \dots$ of r take on two posible values one or two. To decide the value of each digit a_i we consider the corresponding real number r_i in our list.

If the first digit a_{11} of r_1 is not one then we let the first digit a_1 of r be one. If the first digit a_{11} of r_1 is one then we let the first digit a_1 of r_1 be two.

If the second digit a_{22} of r_2 is not one then we let the second digit a_2 of r be one. If the first digit a_{22} of r_2 is one then we let the first digit a_2 of r_2 be two.

In general, we define the nth digit a_n of r as follows:

$$a_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1. \end{cases}$$

As f is surjective there is a natural number n such that $f(n) = r$, that is, $r = r_n$. Suppose that $n = 0$. There are two cases. Either $m = 0$ in which case by definition of m, $m_0 \neq 0$. Or $m = 1$ in which case $m_0 = 0$. Either way, $m \neq m_0$, which contradicts the fact that $r = r_0$. Thus $n \neq 1$.

Now suppose that $n > 0$. What is the nth digit a_n of r_n? If the nth digit is 1 then the nth digit of r_n is not equal to one. If the nth digit is 2 then the nth digit of r_n is equal to one. Either way, $a_n \neq a_{nn}$. This contradicts the fact that $r = r_n$.

Thus r does not belong to the list of real numbers. This contradicts the fact that f is surjective. Therefore the reals are uncountable. □

Corollary 18.7. *There are irrational numbers, that is, there are real numbers that are not rational.*

Proof. Consider the inclusion

$$i \colon \mathbb{Q} \longrightarrow \mathbb{R}$$

The rationals are countable and so i is not bijective. As i is injective it is not surjective. Thus there are reals which are not rational. □