17. Counting: infinite sets

We recall the definition of the cardinality:

Definition 17.1. *We say two sets A and B have the same* ***cardinality****, denoted $|A| = |B|$, if there is a bijection $f: A \longrightarrow B$.*

In fact, the original definition said that f is invertible. But now we know that invertible and bijective are the same. So far we have focused our attention on finite sets. We make some basic observations about the cardinality of a set, which justify the use of the equals sign.

Lemma 17.2. *Let A, B and C be three sets.*

(1) $|A| = |A|$.
(2) If $|A| = |B|$ then $|B| = |A|$.
(3) If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$.

Proof. For (1) note that the identity $\mathrm{id}_A: A \longrightarrow A$ is a bijection.

For (2), by assumption $|A| = |B|$. Therefore there is a bijection $f: A \longrightarrow B$. As f is a bijection f is invertible. Let $g: B \longrightarrow A$ be the inverse of f. Note that f is the inverse of g; indeed $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Thus g is invertible. Thus $|B| = |A|$.

For (3), assume that $|A| = |B|$ and $|B| = |C|$. Then we may find bijections $f: A \longrightarrow B$ and $g: B \longrightarrow C$. As $f: A \longrightarrow B$ is a bijection and $g: B \longrightarrow C$ is a bijection then $g \circ f: A \longrightarrow C$ is a bijection. But then $|A| = |C|$. □

Example 17.3. *The even integers and odd integers have the same cardinality.*

One way to show this is simply to show that the even integers have the same cardinality as the integers and the odd integers have the same cardinality as the integers and then use (17.2).

Let

$$E = \{\, n \in \mathbb{Z} \,|\, n \text{ is even} \,\} \qquad \text{and} \qquad O = \{\, n \in \mathbb{Z} \,|\, n \text{ is odd} \,\}.$$

Define a map

$$f: \mathbb{Z} \longrightarrow E \qquad \text{by the rule} \qquad f(k) = 2k.$$

We check that f is a bijection. Suppose that $n \in E$. Then n is an even integer and so we can find an integer k such that $n = 2k$. But then $f(k) = n$ and so f is surjective.

Now we check that f is injective. Suppose that $f(k_1) = f(k_2)$. Then $2k_1 = 2k_2$. But then $2(k_1 - k_2) = 0$. But then $k_1 - k_2 = 0$ and so $k_1 = k_2$. Thus f is injective.

It follows that f is a bijection and so $|\mathbb{Z}| = |E|$.

Define a map

$$f\colon \mathbb{Z} \longrightarrow O \qquad \text{by the rule} \qquad f(k) = 2k + 1.$$

We check that f is a bijection. Suppose that $n \in O$. Then n is an odd integer and so we can find an integer k such that $n = 2k+1$. But then $f(k) = n$ and so f is surjective.

Now we check that f is injective. Suppose that $f(k_1) = f(k_2)$. Then $2k_1 + 1 = 2k_2 + 1$ so that $2k_1 = 2k_2$. But then $2(k_1 - k_2) = 0$. But then $k_1 - k_2 = 0$ and so $k_1 = k_2$. Thus f is injective.

It follows that f is a bijection and so $|\mathbb{Z}| = |O|$. In particular $|E| = |O|$.

We can also exhibit an explicit bijection

$$f\colon E \longrightarrow O \qquad \text{by the rule} \qquad f(n) = n + 1.$$

We check that f is a bijection. Suppose that $m \in O$. Then m is an odd integer and so we can find an even integer n such that $m = n + 1$. But then $f(n) = m$ and so f is surjective.

Now we check that f is injective. Suppose that $f(n_1) = f(n_2)$. Then $n_1 + 1 = n_2 + 1$ so that $n_1 = n_2$. Thus f is injective. It follows that f is a bijection and so $|E| = |O|$.

Definition 17.4. *Let A be a set. We say that A is* ***countable*** *if either A is finite or $|A|$ has the same cardinality as the integers.*

So far we have seen that the integers, the odd integers and the even integer are infinite countable sets.

Lemma 17.5. *The natural numbers and the positive integers have the same cardinality.*

Proof. Let P be the set of positive integers. Define

$$f\colon \mathbb{N} \longrightarrow P \qquad \text{by the rule} \qquad f(n) = n + 1.$$

We check that f is bijection. We check that f is injective. Suppose that $f(n_1) = f(n_2)$. Then $n_1+1 = n_2+1$. But then $n_1 = n_2$. Therefore f is injective. Now we check that f is surjective. Suppose that $p \in P$. Then P is a positive integer, so that $p \geq 1$. It follows that $p - 1 \geq 0$. Let $n = p - 1$. Then $f(n) = n + 1 = p$. Thus f is surjective. Thus f is a bijection. $\square$

There is a curious feature of (17.5). Suppose we modify the function f defined in (17.5) in a very small way. Let

$$h\colon \mathbb{N} \longrightarrow \mathbb{N}$$

be the function $h(n) = n + 1$. Then h is injective but not surjective. There is nothing mapping to zero. One way to picture all of this is in terms of a hotel. Suppose someone arrives at the hotel and asks if there is a vacancy. The receptionist replies there is always a vacancy, even if the hotel is fully occupied. The person in room zero moves to room one, in room two to room three, and so on, ad infinitum.

Lemma 17.6. *The natural numbers are countable.*

Proof. Define

$$f : \mathbb{Z} \longrightarrow \mathbb{N}$$

by the rule

$$f(n) = \begin{cases} 2n & \text{if } n \text{ is non-negative} \\ -1 - 2n & \text{if } n \text{ is negative.} \end{cases}$$

We check that f is a bijection. We first check that f is surjective. Suppose that $m \in \mathbb{N}$. There are two cases. If m is even then we may find n such that $m = 2n$. Note that $n \geq 0$ so that $f(n) = 2n = m$. If m is odd then $m = 2k + 1 = 2(k + 1) - 1$. Let $n = -k - 1$. As $m \geq 1$, $k \geq 0$ and so $n < 0$. Thus $f(n) = -1 - 2n = 2(k + 1) - 1 = m$. Either way, we may find n such that $f(n) = m$ and so f is surjective.

Now we check that f is injective. Suppose that $f(n_1) = f(n_2)$. There are two cases. Suppose that $n_1 \geq 0$. Then $f(n_1) = 2n_1 = m$ is even. If $n_2 < 0$ then $f(n_2)$ is odd and so $n_2 \geq 0$. But then $2n_1 = 2n_2$ and so $n_1 = n_2$. Now suppose that $n_1 < 0$. Then $m = f(n_1) = -1 - 2n_1$ is odd. If $n_2 \geq 0$ then $f(n_2)$ is even and so $n_2 < 0$. But then $-1 - 2n_1 = -1 - 2n_2$ and so $n_1 = n_2$. Either way, $n_1 = n_2$ and so f is injective. Thus f is a bijection. $\square$