15. Injective surjective and bijective

The notion of an invertible function is very important and we would like to break up the property of being invertible into pieces.

Definition 15.1. *Let $f\colon A \longrightarrow B$ be a function.*

We say that f is ***injective*** *if whenever $f(a_1) = f(a_2)$, for some a_1 and $a_2 \in A$, then $a_1 = a_2$.*

To help understand any property, it sometime helps to write down the simplest example of something that does not have the property. What is the simplest example of a function which is not injective?

Can we take A to be the emptyset? Well, no. In terms of quantifiers, injective is the property:

$$\forall a_1 \in A,\ \forall a_2 \in A,\ (f(a_1) = f(a_2)) \implies (a_1 = a_2).$$

If we negate this then we get

$$\exists a_1 \in A,\ \exists a_2 \in A,\ (f(a_1) = f(a_2)) \wedge (a_1 \neq a_2).$$

The emptyset has no elements and so any function from the emptyset is always injective. How about if A has just one element? We still can not find a function which is not injective. We want to find $a_1 \neq a_2 \in A$, not possible if A has only one element.

But we are okay if A has two elements,

$$A = \{\, \alpha_1, \alpha_2 \,\}.$$

How many elements does B have to have? At least one, since A is not empty. If

$$B = \{\, \beta \,\}$$

then there is one function f from A to B, the constant function which sends everything to β. So f is not injective, since

$$f(\alpha_1) = \beta = f(\alpha_2) \qquad \text{and yet} \qquad \alpha_1 \neq \alpha_2.$$

This is a minimal example of function which is not injective.

One way to think of injective functions is that if f is injective we don't lose any information. If f translates English words into French words, it will be injective provided different words in English get translated into different words in French. If we translate both the word "big" and "large" into the single French word "grand" then we have lost information. If I ask you what word in English corresponds to the word "grand", you would have to say that is depends, it might either be "big" or "large". A function is injective if we can recover a simply by knowing its image $f(a)$.

Another way to describe an injective function is that there is no duplication.

Note that if A is a finite set and $f\colon A \longrightarrow B$ is injective then

$$|A| \leq |B|.$$

There is a companion notion to injective:

Definition 15.2. *Let $f\colon A \longrightarrow B$ be a function. We say f is **surjective** if for every element b of B there is an element a of A mapping to b.*

Again, how small an example can we construct of a function which is not surjective? Now we focus on B. Can we take B to be the emptyset? Well, no. In terms of quantifiers, surjective is the property:

$$\forall b \in B,\ \exists a \in A,\ f(a) = b.$$

If we negate this then we get

$$\exists b \in B,\ \forall a \in A,\ f(a) \neq b.$$

If B is empty then we don't get past the first quantifier.

How about if B contains one element? If we take A to be the emptyset then we win.

Perhaps a more convincing example is if we take A to have one element and B to have two elements,

$$A = \{\, \alpha \,\} \qquad \text{and} \qquad B = \{\, \beta_1, \beta_2 \,\}.$$

In this case we could take f to be the function $f(\alpha) = \beta_1$. Then f is not surjective as β_2 is not the image of any element of A. Of course β_1 is in the image.

In terms of French and English, translation from English into French is surjective, if for every word in French, there is a word in English which we would translate into that word.

Another way to describe a surjective function is that nothing is overlooked.

Note that if B is a finite set and $f\colon A \longrightarrow B$ is surjective then

$$|A| \geq |B|.$$

Definition 15.3. *A function f is a **bijection** (or f is **bijective**) if it is injective and surjective.*

The main point of all of this is:

Theorem 15.4. *Let $f\colon A \longrightarrow B$ be a function.*

Then f is invertible if and only if it is bijective.

Proof. Suppose that f is invertible. We have to show that f is bijective.

Let $g\colon B \longrightarrow A$ be the inverse of f. We check that f is injective. Suppose that $f(a_1) = f(a_2)$ for some a_1 and $a_2 \in A$. We have $g \circ f = \mathrm{id}_A$. If we apply g to both sides we get

$$\begin{aligned} a_1 &= \mathrm{id}_A(a_1) \\ &= (g \circ f)(a_1) \\ &= g(f(a_1)) \\ &= g(f(a_2)) \\ &= (g \circ f)(a_2) \\ &= \mathrm{id}_A(a_2) \\ &= a_2. \end{aligned}$$

Thus f is injective.

Now we check that f is surjective. Pick $b \in B$. Let $a = g(b)$. We have $f \circ g = \mathrm{id}_B$. It follows that

$$\begin{aligned} f(a) &= f(g(b)) \\ &= (f \circ g)(b) \\ &= \mathrm{id}_B(b) \\ &= b. \end{aligned}$$

Thus f is surjective. Thus we have shown that if f is invertible then f is bijective.

Now suppose that f is bijective. We have to show that f is invertible. Define a function $g\colon B \longrightarrow A$ as follows. Given $b \in B$, as f is surjective, we may find $a \in A$ such that $f(a) = b$. Note that a is unique, as f is injective.

We check that g is the inverse of f. We first check that $g \circ f = \mathrm{id}_A$. Both sides of this equation are functions from A to A. Therefore it suffices to check that they have the same effect on an element a of A. Note that if $b = f(a)$ then $g(b) = a$, by definition of g. We have

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) \\ &= a \\ &= \mathrm{id}_A(a), \end{aligned}$$

where we used the observation above to get from line one to line two. Thus $g \circ f = \mathrm{id}_A$.

Now we check that $f \circ g = \mathrm{id}_B$. Both sides of this equation are functions from B to B. Therefore it suffices to check that they have the same effect on an element b of B. Let $a = g(b)$. Then by definition of g, we have $f(a) = b$. We have

$$\begin{aligned}(f \circ g)(b) &= f(g(b)) \\ &= b \\ &= \mathrm{id}_A(b),\end{aligned}$$

where we used the observation above to get from line one to line two. Thus $f \circ g = \mathrm{id}_B$.

It follows that g is the inverse of f. Thus we have shown that if f is bijective then f is invertible. □

Note that it is not enough to construct a one-sided inverse to conclude that f is invertible, or equivalently, bijective.

Example 15.5. *Suppose we start with the quintessential example of a function $f\colon A \longrightarrow B$ which is surjective but not injective.*

$$A = \{\, \alpha_1, \alpha_2 \,\} \qquad \text{and} \qquad B = \{\, \beta \,\}.$$

and f is the constant function which sends everything to β. Let $g\colon B \longrightarrow A$ be the function $g(\beta) = \alpha_1$. Then $f \circ g = \mathrm{id}_B\colon B \longrightarrow B$. But $g \circ f\colon A \longrightarrow A$ is a contsant function, which sends everything to α_1.

Example 15.6. *Now consider the quintessential example of a function $f\colon A \longrightarrow B$ which is injective but not surjective.*

We have

$$A = \{\, \alpha \,\} \qquad \text{and} \qquad B = \{\, \beta_1, \beta_2 \,\}.$$

and $f(\alpha) = \beta_1$. Let $g\colon B \longrightarrow A$ be the constant function which sends everything to α. Then $g \circ f = \mathrm{id}_A\colon A \longrightarrow A$ but $f \circ g\colon B \longrightarrow B$ is the constant function to β_1.

Lemma 15.7. *Let $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ and $h\colon C \longrightarrow D$ be three functions.*

Then we have an equality of functions:

$$h \circ (g \circ f) = (h \circ g) \circ f\colon A \longrightarrow D.$$

Proof. Both sides are functions from A to D. Therefore it suffices to check they have the same effect on an arbitrary element a of A.

$$\begin{aligned}h \circ (g \circ f)(a) &= h((g \circ f)(a)) \\ &= h(g(f(a))) \\ &= (h \circ g)(f(a)) \\ &= ((h \circ g) \circ f)(a).\end{aligned}$$

Thus $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are the same function. $\square$

Lemma 15.8. *If $f\colon A \longrightarrow B$ is an invertible function then the inverse is unique.*

Proof. Suppose that $g_1\colon B \longrightarrow A$ and $g_2\colon B \longrightarrow A$ are both inverses of f. Consider the product $g_1 \circ f \circ g_2$. Note that, by (15.7), we don't have to be careful to correctly bracket this expression.

On the one hand, we have

$$\begin{aligned} g_1 \circ f \circ g_2 &= (g_1 \circ f) \circ g_2 \\ &= \mathrm{id}_A \circ g_2 \\ &= g_2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g_1 \circ f \circ g_2 &= g_1 \circ (f \circ g_2) \\ &= g_1 \circ \mathrm{id}_B \\ &= g_1. \end{aligned}$$

Thus $g_1 = g_2$. $\square$