13. Functions

Definition 13.1. *Let A and B be two sets. The* ***Cartesian product*** *of A and B, denoted $A \times B$, is the set of all ordered pairs*

$$A \times B = \{ (a, b) \mid a \in A, b \in B \}.$$

Remark 13.2. *Again for the sake of completeness, observe that the formal way to define an ordered pair using the language of sets is*

$$(a, b) = \{ \{ a \}, \{ a, b \} \}.$$

We have already seen unordered pairs in the context of edges of a graph:

$$ab = \{ a, b \}.$$

Example 13.3. *Let*

$$A = \{ a, b, c \} \qquad \text{and} \qquad B = \{ c, d \}$$

Then

$$A \times B = \{ (a, c), (a, d), (b, c), (b, d), (c, c), (c, d) \}.$$

Note that if A and B are finite sets then

$$|A \times B| = |A| \cdot |B|.$$

We now introduce probably the most important concept in mathematics.

Definition 13.4. *A* ***function*** *between two sets A and B, denoted $f \colon A \longrightarrow B$, is a subset $\Gamma_f \subset A \times B$ such that for every element a of A, there is a unique element $b \in B$ such that $(a, b) \in \Gamma_f$,*

$$\forall a \in A,\ \exists! b \in B,\ (a, b) \in \Gamma_f.$$

A is called the ***domain*** *and B is called the* ***range****.*

$\exists! b \in B$ means that there is a unique element with the given property.

In practice, we will use the notation $f(a) = b$, so that $(a, f(a)) \in \Gamma_f$. The notion of a function is almost as basic as that of a set.

We start with a degenerate example. Suppose that $A = \emptyset$. If B is any set then there is one function f from A to B,

$$\Gamma_f = \emptyset \subset \emptyset \times B.$$

We are supposed to quantify over A. As A is empty there is nothing to check.

Now suppose that $B = \emptyset$ and A is any set. Is there a function from f from A to B? No, unless $A = \emptyset$. If A is not empty then we have to find an element of the emptyset with a certain property, which is not possible.

If B has one element, $B = \{ b \}$, there is always exactly one function from A to B. We just send every element of A to B.

Definition 13.5. *Let A be any set and let B be a set and let b be an element of B, $b \in B$. Define a function by $f \colon A \longrightarrow B$ by*

$$\Gamma_f = \{ (a, b) \mid a \in A \} = A \times \{ b \} \subset A \times B.$$

Using the more standard notation, we have

$$f(a) = b \qquad \text{for all} \qquad a \in A.$$

For obvious reasons we call any such function a constant function.

Definition 13.6. *If A is any set, the* ***identity function*** *from A to A, denoted $\mathrm{id}_A \colon A \longrightarrow A$, is the function*

$$\{ (a, a) \mid a \in A \} \subset A \times A.$$

Using the more standard notation, we have

$$\mathrm{id}_A(a) = a.$$

More generally, we have

Definition 13.7. *Let A be a subset of B, $A \subset B$. The* ***inclusion*** *of A into B, denoted $i_A \colon A \longrightarrow B$ is the function*

$$\{ (a, a) \mid a \in A \} \subset A \times B.$$

Using the more standard notation, we have

$$i_A(a) = a.$$

Note that the identity function is a special case of an inclusion, where $B = A$.

Example 13.8. *Let*

$$A = \{ a, b, c \} \qquad \textit{and} \qquad B = \{ c, d \}$$

Then

$$\Gamma = \{ (a, c), (b, d), (c, c) \},$$

is a function $f \colon A \longrightarrow B$. Using more standard notation, we have

$$f(a) = c, \qquad f(b) = d \qquad \text{and} \qquad f(c) = c.$$

Let X be a finite set. There is a function from the powerset of X to the natural numbers:

$$f \colon \wp(X) \longrightarrow \mathbb{N} \qquad \text{given by} \qquad f(A) = |A|.$$

An element of the powerset, is a subset of X. As X is finite, this subset has finite cardinality and f just assigns to A its cardinality.

If

$$X = \{ a, b, c, d \}$$

then

$$f(\{ a \}) = 1 \qquad \text{and} \qquad f(\{ b, d \}) = 2.$$

Suppose that $G = (V, E)$. We have already implicitly defined a function

$$d \colon V \longrightarrow \mathbb{N} \qquad \text{given by} \qquad d(v) = \text{the degree of } v.$$

If X is a set and A is a subset, we can define a function

$$\chi_A \colon X \longrightarrow \{ 0, 1 \} \qquad \text{given by} \qquad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

If

$$X = \{ a, b, c, d \} \qquad \text{and} \qquad A = \{ a \}$$

then

$$\chi_A(a) = 1 \qquad \text{but} \qquad \chi_A(c) = 0.$$

χ_A is called the **characteristic function** of A.

Definition 13.9. *A **sequence** is a function* $f \colon \mathbb{N} \longrightarrow A$.

Of course one way to denote a sequence is to write it out:

$$a_0, a_1, a_2, \dots$$

The Fibonacci numbers are a sequence of natural numbers.