12. Quantifiers

All men are mortal
Socrates is a man
Therefore, Socrates is mortal.

Some truths are unfortunately undeniable.

Mathematicians like universal statements.

> The square of *every* even integer is even.
>
> The solution of *any* quadratic equation is given by the quadratic formula.
>
> For *every* odd integer n there *exists* an integer k such that $n = 2k + 1$.

It is sometimes convenient to make this more explicit using quantifiers, for all and there exists:

$$\forall x \in A,\ P(x) \qquad \text{and} \qquad \exists x \in A,\ P(x).$$

The first reads for every $x \in A$, $P(x)$ holds and the second reads, there is an element x of A such that $P(x)$ holds.

Note one special choice of A, the emptyset.

$$\forall x \in \emptyset,\ P(x)$$

is always true;

$$\forall x \in \emptyset,\ x \text{ is odd} \qquad \text{and} \qquad \forall x \in \emptyset,\ x \text{ is even}$$

are both true. The point is that the emptyset has no elements and so there is nothing to check. Conversely

$$\exists x \in \emptyset,\ P(x)$$

is always false;

$$\exists x \in \emptyset,\ x \text{ is odd} \qquad \text{and} \qquad \exists x \in \emptyset,\ x \text{ is even}$$

are both false. The point is that the emptyset has no elements and so there is no way to find an element of the emptyset with the correct properties.

It is useful to know how to combine and negate quantifiers.

Let's start with a well-known example:

> An apple a day keeps the doctor away.

How would we translate this into statements using quantifiers? We are quantifying over all days.

> For every (all) day(s) (there exists) an apple keeps the doctor away.

The key thing to realise is that the order of the quantifiers is crucial. If we tried to switch them:

> (there exists) One apple, taken over the course of a lifetime (all days), keeps the doctor away.

An optimistic saying gets replaced by a fantasy.

Not surprisingly it is the same in mathematics; the order and placement of universal and existential quantifiers is crucial to the meaning.

For example, both

$$\forall n \in \mathbb{Z},\ (n \text{ is odd}) \implies \exists k \in \mathbb{Z}\ (n = 2k+1),$$

and

$$\forall n \in \mathbb{Z},\ \exists k \in \mathbb{Z}\ (n \text{ is odd}) \implies (n = 2k+1),$$

are true statements. They are restatements of the result that all odd integers n are of the form $2k+1$. The first is the obvious way to restate the result. The second is also correct though. Imagine if we first pick n and then pick k. We want the implication to come out correctly. If n is odd, say $n = 57$, we take the appropriate value of k, say $k = 28$. If n is even, say $n = 10$, then we take any value of k, say $k = 0$. The statement

$$(n \text{ is odd}) \implies (n = 2k+1),$$

is correct as n is not odd.

On the other hand, if one moves the quantifiers, one gets an absurd statement

$$\exists k \in \mathbb{Z},\ \forall n \in \mathbb{Z}\ (n \text{ is odd}) \implies (n = 2k+1).$$

There is supposed to be one single integer k such that whenever n is odd then $n = 2k+1$. Clearly absurd, since there is only one integer $2k+1$ and infinitely many odd integers.

Note that it does not really matter if you swap universal quantifiers or swap existential quantifiers:

$$\forall n \in \mathbb{Z}, x \in \mathbb{Z}, y \in \mathbb{Z}, z \in \mathbb{Z}\ (n > 2, x > 0, y > 0, z > 0) \implies (x^n + y^n \neq z^n)$$

is a statement of Fermat's last theorem. Re-ordering the variables won't change anything.

The other interesting thing is to negate quantifiers. How do we negate the universal quantifier?

$$\neg(\forall x \in A,\ P(x)).$$

Let's take an example. Suppose that A is a subset of the integers.

$$\forall x \in A,\ x \text{ is even}.$$

This is the assertion that every element of A is even. If this is not true, this means there is an element of A that is odd,

$$\neg(\forall x \in A,\ x \text{ is even}) = \exists x \in A,\ x \text{ is odd}.$$

This is the general rule

$$\neg(\forall x \in A,\ P(x)) = \exists x \in A,\ \neg P(x).$$

Since

$$\neg(\neg P) = P,$$

this means that

$$\neg(\exists x \in A,\ P(x)) = \forall x \in A,\ \neg P(x).$$

Note that this means one can negate very complicated statements, even if one doesn't really understand the meaning. This can occasionally be useful. The rule is very simple. Switch for all's and exist's and negate the statement inside. For example, the negation of

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ (|x - y| < \delta \implies |f(x) - f(y)| < \epsilon)$$

is the statement

$$\exists \epsilon > 0,\ \forall \delta > 0,\ \exists x \in \mathbb{R},\ \exists y \in \mathbb{R},\ (|x - y| < \delta \wedge |f(x) - f(y)| > \epsilon)$$

Here we use the fact that

$$\neg(P \implies Q) = P \wedge \neg Q.$$

Definition 12.1. *Let $x_1, x_2, \dots$ be a sequence of real numbers. We say that the sequence is* ***monotonic increasing*** *if for every n, $a_{n+1} \geq a_n$. We say that the sequence is* ***monotonic decreasing*** *if for every n, $a_{n+1} \leq a_n$. We say that a sequence is* ***monotonic*** *if it is either monotonic increasing or decreasing.*

The sequence

$$1, \quad 2, \quad 3, \quad \dots,$$

is monotonic increasing and the sequence

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \dots$$

is monotonic decreasing. The sequence

$$1, \quad -1, \quad 1, \quad -1, \dots$$

is not monotonic.

Definition 12.2. *Let $A \subset \mathbb{R}$ be a set. We say that A is **bounded from above** if there is a real number U such that if $a \in A$ then $a \leq U$. The **supremum** of A, denoted $\sup A$, is an upper bound which is at most any other upper bound. The **maximum** of A, denoted $\max A$, is an upper bound which belongs to A.*

*We say that A is **bounded from below** if there is a real number L such that if $a \in A$ then $a \geq L$. The **infimum** of A, denoted $\inf A$, is a lower bound which is at least any other lower bound. The **minimum** of A, denoted $\min A$, is a lower bound which belongs to A.*

If one wants to state the property of being bounded above in terms of quantifiers one gets

$$\exists U \in \mathbb{R},\ \forall a \in A,\ a \leq U.$$

Consider trying to swap the quantifiers

$$\forall a \in A,\ \exists U \in \mathbb{R},\ a \leq U.$$

This is a much weaker statement which is true for any set A. Given $a \in A$ just take $U = a$. Then

$$a \leq a = U.$$

If $x_1, x_2, \dots$ is a sequence of real numbers then bounded will refer to the set

$$\{\, x_i \,|\, i \in \mathbb{Z}, i > 0 \,\}.$$

The first sequence is not bounded from above:

Lemma 12.3. *The set $\mathbb{N}$ is not bounded from above.*

Proof. Suppose not, suppose that $\mathbb{N}$ is bounded above by $U \in \mathbb{R}$. Pick $n \in \mathbb{N}$ bigger than U (for example, take the decimal expansion of U, ignore the part after the decimal point and add one). This is a contradiction.

Thus $\mathbb{N}$ is not bounded from above. $\square$

As for the second sequence:

Lemma 12.4. 0 *is the infimum of*

$$A = \{\, \frac{1}{n} \,|\, n \in \mathbb{N} \,\}$$

and A has no minimum.

Proof. If $a \in A$ then

$$a = \frac{1}{n} > 0.$$

Thus 0 is a lower bound.

Suppose that $L > 0$ is a larger lower bound. We will derive a contradiction. Pick

$$n > \frac{1}{L}.$$

It follows that

$$\begin{aligned} L &= L \cdot 1 \\ &= L \cdot \frac{1}{n} \cdot n \\ &> L \cdot \frac{1}{nL} \\ &= \frac{1}{n} \in A. \end{aligned}$$

This contradicts the fact that L is a lower bound.

Thus 0 is the infimum of A. The elements of A are all positive and so no element of A is a lower bound. Thus A has no minimum. □

Remark 12.5. *Note that there is some scratch work that goes into the proof of* (12.4). *We know that to get a contradiction we need to find an element $a \in A$ such that $L > a$. The elements of A are of the form $1/n$. To get $1/n < L$ we need to pick $n > 1/L$, which is where we start with L.*

Theorem 12.6 (Completeness of the real numbers)**.** *Every non-empty set of real numbers A which is bounded from above has a supremum.*

Every non-empty set of real numbers A which is bounded from below has an infimum.

Corollary 12.7. *If $x_1, x_2, \dots$ is a monotonic increasing sequence of real numbers which is bounded from above has a supremum.*

If $x_1, x_2, \dots$ is a monotonic decreasing sequence of real numbers which is bounded from below has an infimum.

In the case of a sequence, we call the infimum or supremum the limit (note that limits exist in much greater generality though).