11. The power set

> Please accept my resignation. I don't want to belong to any club that will accept me as a member.
>
> — Groucho Marx

We introduce one more operation on sets, perhaps the most important one:

Definition 11.1. *Let A be a set. The* ***power set*** *of A, denoted $\wp(A)$, is the set of all subsets of A.*

So the set B belongs to the powerset of A if and only if $B \subset A$. Formally,

$$B \in \wp(A) \iff B \subset A.$$

For example, if

$$A = \{\,\text{R,B,O}\,\}$$

the power set of A is

$$\wp(A) = \{\,\emptyset, \{\,\text{R}\,\}, \{\,\text{B}\,\}, \{\,\text{O}\,\}, \{\,\text{B,O}\,\}, \{\,\text{R,O}\,\}, \{\,\text{R,B}\,\}, \{\,\text{R,B,O}\,\}\,\}.$$

Note the following:

$$\wp(\emptyset) = \{\,\emptyset\,\} \qquad \text{and} \qquad \wp(\{\,a\,\}) = \{\,\emptyset, \{\,a\,\}\,\}.$$

In general, we have $\emptyset \in \wp(A)$ and $A \in \wp(A)$, since $\emptyset \subset A$ and $A \subset A$.

Imagine we have a set with two elements a and b, $A = \{\,a, b\}$. How does one form a subset B of A? Well, for each element of A we have two decisions, put this element into B or not? Put differently, for each element of A, we can either keep that element or throw it away. In our case, we can decide to keep a or throw it away and we can decide to keep b or throw it away.

$$\wp(\{\,a, b\,\}) = \{\,\emptyset, \{\,a\,\}, \{\,b\,\}, \{\,a, b\,\}\,\}.$$

Thinking this way, it is not surprising that the powerset has 4 elements. In general, suppose that A has n elements, $a_1, a_2, \dots, a_n$,

$$A = \{\,a_i \,|\, 1 \leq i \leq n\,\}.$$

To form an element of the powerset, that is, a subset B of A, we just have to decide what to include (or what to exclude). For each i, we have a choice. Keep a_i or throw it away. This gives us two choices for each i and in total we have 2^n choices.

Putting all this together, we have

$$|A| = n \implies |\wp(A)| = 2^n = 2^{|A|}.$$

The form of set theory presented in Math 109 was essentially the work of a German mathematician, logician and philosopher, Frege, with one small important twist.

Frege proposed that a set is any collection of objects with a property, or predicate $P(x)$.

Frege's definition of a set

A set A is any collection of objects that satisfies a property

$$A = \{\, x \mid P(x) \,\}.$$

Frege was putting the final touches to his theory when he received a letter from an undergraduate, Bertrand Russell, who asked him what happens when you consider the following predicate:

$$P(x) = \text{ x is not an element of itself } = x \notin x.$$

Let A be the set of all sets which are not elements of themselves,

$$A = \{\, x \mid x \notin x \,\}.$$

Question 11.2. *Is A an element of itself?*

The statement $A \in A$ is either true or false.

Let's suppose it is true. Then $P(A)$ is false, since $P(x)$ is nothing but the negation of $x \in x$. But then,

$$A \notin \{\, x \mid x \notin x \,\} = A.$$

by definition of A, that is, $A \notin A$. Therefore the statement $A \in A$ is false. This is a contradiction.

Okay, so let's suppose the statement $A \in A$ is false. In this case $A \notin A$ and so $P(A)$ is true. But then,

$$A \in \{\, x \mid x \notin x \,\} = A.$$

by definition of A, that is, $A \in A$. Therefore the statement $A \notin A$ is false. This is also a contradiction.

It doesn't seem there is any easy way out of this. This problem is now known as *Russell's paradox*. It is understood that one needs to impose some sort of extra axioms to avoid this problem. Russell and Whitehead pursued a theory of types; this was never really popular in mathematics, but it is quite popular in theoretical computer science. Mathematicians get around Russell's paradox by throwing in the axiom of foundation.

Axiom 11.3 (Axiom of Foundation). *For every set x we can find an element y of x such that $y \cap x = \emptyset$.*

(We state this axiom for completeness). In particular no set is a member of itself and we are only allowed to specify subsets of sets using predicates. One way to view Russell's paradox is that we are not allowed to call a collection a set if it is too big. The axioms allow you to create bigger sets, but in a controlled way. In this sense, the power set is one of the most interesting axioms, since it the only axiom which creates truly bigger sets. For example the axiom of foundation and the existence of power sets, says there is no set which contains all other sets.

Using set theory, we are supposed to be able to model all of mathematics. We give a simple example. How do we construct the natural numbers, the non-negative integers

$$\mathbb{N} = \{\, n \in \mathbb{Z} \,|\, n \geq 0 \,\}?$$

Well, let's start with zero. It is sort of obvious that zero has to be the emptyset:

$$0 = \emptyset.$$

In fact it would be nice if the set n contains precisely n elements.

Definition 11.4. *Let x be a set. x^+ denotes the set whose elements are x and the elements of x:*

$$x^+ = \{\, x \,\} \cup x = \{\, y \,|\, (y = x) \vee (y \in x) \,\}.$$

We define 1 to be the successor of 0,

$$\begin{aligned} 1 &= 0^+ \\ &= \emptyset^+ \\ &= \{\, y \,|\, (y = \emptyset) \vee (y \in \emptyset) \,\} \\ &= \{\, \emptyset \,\} \\ &= \{\, 0 \,\}. \end{aligned}$$

It is pretty obvious how to continue

$$\begin{aligned} 2 &= 1^+ \\ &= \{\, 0 \,\}^+ \\ &= \{\, y \,|\, (y = 1) \vee (y \in \{\, 0 \,\}) \,\} \\ &= \{\, 0, 1 \,\} \\ &= \{\, \emptyset, \{\, \emptyset \,\} \,\}. \end{aligned}$$

$$\begin{aligned}3 &= 2^+\\ &= \{\,0,1\,\}^+\\ &= \{\,y \mid (y=2) \vee (y \in \{\,0,1\,\})\,\}\\ &= \{\,0,1,2\,\}\\ &= \{\,\emptyset, \{\,\emptyset\,\}, \{\,\emptyset, \{\,\emptyset\,\}\,\}\,\}.\end{aligned}$$

In general,

$$\begin{aligned}n+1 &= n^+\\ &= \{\,0,1,2,\dots,n-1\,\}^+\\ &= \{\,y \mid (y=n) \vee (y \in \{\,0,1,2,\dots,n-1\,\})\,\}\\ &= \{\,0,1,2,3,\dots,n\,\}.\end{aligned}$$

Notice how the natural number n, when considered as a set, has the nice property that it contains n elements.

We introduce one more axiom, called the axiom of infinity:

Axiom 11.5 (Axiom of Infinity)**.** *There is a set I which contains the empty set and has the property that if $x \in I$ then so is x^+, $x^+ \in I$.*

As the name might suggest, the axiom of infinity guarantees that there are infinite sets. Using the axiom of infinity, one can construct the natural numbers:

$$\omega = \{\,0,1,2,3,\dots\,\}.$$

What does the powerset of ω look like? To construct an element of the powerset, we need to construct a subset of ω. For each natural number we have a choice, to include this number or not. Let's record this choice, using 0 to indicate rejection and 1 to indicate acceptance. Now order this sequence of 0's and 1's into a string and put a decimal point at the front:

$$.000111010100000011100011101 01\dots.$$

The first three zeroes indicate that our subset does not contain 0, 1, or 2, the next three ones indicate our subset does contain 3, 4, and 5 and so on.

Now think of this string as a number in binary. We get a real number between zero and one, an element of the closed interval

$$[0,1] = \{\,r \in \mathbb{R} \mid 0 \leq r \leq 1\,\}.$$

Thus the powerset of ω is essentially in correspondence with the closed interval $[0,1]$ (one has to be a little bit careful about recurring numbers. The subset ω will give rise to the binary numbers with all 1's which corresponds to the real number $1 = 0.111111\ldots$.)