10. Set theory

We now introduce set theory, which is the language used to describe all of mathematics.

We start with the notion of a set. A set is a collection of objects. One way to describe a set is just to list its objects:

$$A = \{ \text{red, blue, orange} \}, \qquad \text{and} \qquad B = \{ \text{chalk, cheese, } 1, 2, \pi \}.$$

A is the set with three objects, the colours, red, blue and orange. Note how you are allowed to mix chalk, cheese and numbers.

Given an object, a, one can ask if a belongs to A, or if a is an element of A.

$$a \in A$$

(read, a belongs to A, or a is an element of A) is a proposition, which is true if a belongs to A and false otherwise.

Thus red $\in A$ and $\pi \in B$ are both true, but $e \in B$ is false.

$$a \notin A$$

(read a does not belong to A, a is not an element of A), is shorthand for the proposition $\neg(a \in A)$. Thus red $\notin A$ and $\pi \notin B$ are both false, but $e \notin B$ is true.

Note that sets can even contain other sets,

$$C = \{ 1, \{ 1 \} \}, \qquad \text{and} \qquad D = \{ A, B, C, \text{apples, oranges} \}.$$

C contains two elements, 1 and the set that contains 1. Or if you like C is a box with two elements. 1 and another box that contains 1.

D is even worse; it contains five elements, the three sets A, B and C, and two other elements, apples and oranges.

We can't really define what a collection is, but we can say when two sets are equal:

Definition 10.1. *Two sets A and B are **equal**, denoted $A = B$, if and only if they have the same elements.*

Formally,

$$A = B \qquad \Longleftrightarrow \qquad (x \in A \iff x \in B).$$

One reason definitions are useful is to remove ambiguity: the following two sets

$$\{ 1 \} = \{ 1, 1 \}$$

are equal as they have the same elements. Namely they both only contain the element 1, no matter that in the second set, it looks as though 1 is repeated. Note the difference between this set and C, which contains two elements. Note also that

$$\{ \text{red, blue, orange} \} = \{ \text{blue, orange, red} \}.$$

We will give a working definition of the cardinality, which we will revisit later:

If A is a set, the cardinality of A, denoted $|A|$, is

$$|A| = \begin{cases} n & \text{if } A \text{ has finitely many elements.} \\ \infty & \text{otherwise.} \end{cases}$$

where n is the number of elements of A.

There is a slightly weaker notion than equality.

Definition 10.2. *We say that A is a **subset** of B, denoted $A \subset B$, if every element of A is an element of B.*

Formally,

$$A \subset B \qquad \iff \qquad (x \in A \implies x \in B).$$

For example,

$$\{ \text{red} \} \subset \{ \text{red, blue, orange} \} \qquad \text{and} \qquad \{ \text{red, orange} \} \subset \{ \text{red, blue, orange} \}.$$

Note that however that the set $\{ \text{red} \}$ is not an element of the set $\{ \text{red, blue, orange} \}$ and that the object red is not a subset of the set $\{ \text{red, blue, orange} \}$.

Note also that any set A is always a subset of itself, $A \subset A$.

The following sets are very useful, the set of all integers $\mathbb{Z}$, the set of all rational numbers, the set of all real numbers $\mathbb{R}$ and the set of all complex numbers $\mathbb{C}$. We have

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C},$$

that is, every integer is a rational number, every rational number is a real number, and every real number is a complex number.

The following is really easy but it is one of the most used results in all of mathematics:

Lemma 10.3. *Let A and B be two sets.*

The following are equivalent:

(1) $A = B$.
(2) $A \subset B$ and $B \subset A$.

Proof. Suppose that (1) is true. If $x \in A$ then $x \in B$ and so $A \subset B$. By symmetry $B \subset A$. Thus (2) is true.

Now suppose that (2) is true. Suppose that $x \in A$. As $A \subset B$ then $x \in B$. Suppose that $x \in B$. As $B \subset A$ then $x \in A$. Thus $A = B$ and so (1) is true. $\square$

It is often convenient to define a set by a property (a predicate):

$$B = \{\, x \in A \,|\, P(x) \,\}.$$

The elements of B are the elements of A which satisfy the predicate $P(x)$.

For example,

$$B = \{\, x \in \mathbb{Z} \,|\, x \text{ is even} \,\} = \{\, -\dots, -4, -2, 0, 2, 4, 6, \dots, \,\}.$$

are the even integers,

$$\begin{aligned} R &= \{\, c \,|\, c \text{ is a colour of the rainbow} \,\} \\ &= \{\, \text{red, orange, yellow, green, blue, indigo, violet} \,\}. \end{aligned}$$

There is one particularly interesting predicate

Definition 10.4. *Let A be any set. The* ***emptyset****, denoted $\emptyset$, is the subset of all elements of A not equal to themselves.*

Formally,

$$\emptyset = \{\, x \in A \,|\, x \neq x \,\}.$$

The emptyset is characterised by the property that it has no elements. The emptyset is a subset of every set. If A is a set then we have to check that every element of the emptyset is an element of A. But the emptyset has no elements, so this is vacuously true.

One way to denote the emptyset is

$$\emptyset = \{\,\}.$$

This notation emphasises the fact that the emptyset has no elements. The emptyset is like a box with no elements.

Notice the difference between

$$\{\,\} \qquad \text{and} \qquad \{\,\{\,\}\,\}.$$

The first is the emptyset and the second is a set which contains the emptyset (a box which contains an empty box).

There are operations to create sets from other sets:

Definition 10.5. *If A and B are two sets, the* ***intersection*** *of A and B, denoted $A \cap B$, is the set of all elements common to A and B.*

Formally

$$x \in A \cap B \iff (x \in A) \wedge (x \in B).$$

Equivalently, we can define

$$A \cap B = \{\, a \in A \,|\, a \in B \,\} = \{\, b \in B \,|\, b \in A \,\}.$$

Note that $A \cap B \subset A$ and $A \cap B \subset B$.

Definition 10.6. *If A and B are two sets, the* ***union*** *of A and B, denoted $A \cup B$, is the set of all elements which belong to either A or B.*

Formally

$$x \in A \cup B \iff (x \in A) \vee (x \in B).$$

Note that $A \subset A \cup B$ and $B \subset A \cup B$.

Definition 10.7. *If A and B are two sets, the* ***difference*** *of A and B, denoted $A \setminus B$, is the set of all elements of A which are not elements of B.*

Formally,

$$x \in A \setminus B \qquad \iff \qquad (x \in A \wedge x \notin B).$$

We could also define the difference as

$$A \setminus B = \{\, a \in A \,|\, a \notin B \,\}.$$

If B is a subset of A then the difference is also known as the **complement** of B in A. If $A = \mathbb{Z}$, the integers and B is the set of even integers then $A \setminus B$ is the set of odd integers, the complement of the even integers. If R is the set of colours of the rainbow and

$$S = \{\, \text{red, blue, orange} \,\}, \qquad \text{then} \qquad R \setminus S = \{\, \text{yellow, green, indigo, violet} \,\}.$$

Definition 10.8. *If A and B are two sets, the* ***symmetric difference*** *of A and B, denoted $A \triangle B$, is the set*

$$(A \setminus B) \cup (B \setminus A).$$

Formally,

$$x \in (A \setminus B) \cup (B \setminus A) \iff (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A).$$

Lemma 10.9. *If A and B are two sets then*

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Proof. We want to prove that two sets are equal,

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

We prove that there is an inclusion both ways. We first show that the LHS is a subset of the RHS.

Suppose that $x \in (A \setminus B) \cup (B \setminus A)$. Then either $x \in (A \setminus B)$ or $x \in (B \setminus A)$. Suppose that $x \in (A \setminus B)$. It follows that $x \in A$ and $x \notin B$. As $x \in A$ it follows that $x \in A \cup B$. As $x \notin B$ it follows that $x \notin A \cap B$. Therefore $x \in (A \cup B) \setminus (A \cap B)$. By symmetry if $x \in (B \setminus A)$ then $x \in (A \cup B) \setminus (A \cap B)$. Thus the LHS is a subset of the RHS.

Now we show that the RHS is a subset of the LHS. Suppose that $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in (A \cup B)$ but $x \notin (A \cap B)$. As $x \in (A \cup B)$ it follows that $x \in A$ or $x \in B$. Suppose that $x \in A$. As $x \notin A \cap B$ and $x \in A$ it follows that $x \notin B$. But then $x \in B \setminus A$ and so $x \in A \triangle B$. By symmetry if $x \in B$ then $x \in A \triangle B$. Thus the RHS is a subset of the LHS.

As we have an inclusion both ways, we have

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B). \qquad \square$$