> Lasciate ogne speranza, voi ch'intrate
>
> — Dante Alighieri

1. A little logic: propositions

> Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true
>
> — Bertrand Russell

This course is concerned with how to present mathematical arguments. This involves the language to express an argument, the underlying logic behind the argument, the best way to structure the argument and picking the appropriate hypothesis. A mathematical argument is really a conversation between two people, the writer and the reader. What to say and how to say it depends a lot on your audience.

By analogy with learning how to write, learning how to write down mathematical proofs involves reading lots of different mathematical proofs and lots of practice writing down your own.

We start with a little bit of logic. Mathematical statements are propositions. A proposition is a sentence which is either true or false. Examples are

Socrates is a man.

The rain in Spain falls mainly on the plain.

An apple a day keeps the doctor away.

He has fallen in the water.

This is a pipe.

The square of three is nine.

The are infinitely many twin primes.

The sun is the same as the moon.

$2 + 2 = 5$.

The last two statements are propositions, even if they are false.

The following are not propositions:

What time is it, Eccles?

This is a question, not a true or false statement.

Look, over there, a tiger!

This is a warning.

Don't walk on the grass.

This is an admonition.

The following is not a proposition,

$x = 2$

unless we know more about x. As it stands it is neither true or false. x might be equal to 2, but it might be equal to 3 or 4, and so on.

The **negation** of a proposition reverses the truth value. The negation of

This is a pipe.

is

This is not a pipe.

If we use the sympol p to denote a proposition then $\neg p$ denotes the negation of p. $\neg p$ is false if p is true and $\neg p$ is true if p is false.

Is the following a proposition?

I am a false statement.

It has the grammatical structure of a statement. Let's suppose this is a proposition. Call it p. An equivalent formulation of p is:

p is false.

Is p true or false? If p is true then p says it is false, nonsense. Okay, so then p must be false. In this case $\neg p$ is true, that is,

p is true.

which is again nonsense. So p is not a proposition. This example seems a bit worrying. It looks like we can escape this sort of paradox by not allowing propositions to refer to themselves.

Okay, so what about the pair of statements:

p: q is true. q: p is false.

After a little bit of thought we realise we run into exactly the same problem as before.

There are a couple of interesting ways to combine propositions. If p and q are propositions then $p \wedge q$, read p and q, is the proposition which is only true if p and q are both true; otherwise it is false.

The rain in Spain falls mainly on the plain and this is a pipe.

is the conjunction of

The rain in Spain falls mainly on the plain.

and

This is a pipe.

For example

$$(3^2 = 9) \wedge (2 + 2 = 5),$$

is false, because the second term is false.

We can also form the proposition $p \vee q$, read p or q. It is true if either p is true or q is true or both p and q are true. For example

$$(3^2 = 9) \vee (2 + 2 = 5),$$

is true. Note also that

$$(3^2 = 9) \vee (2 + 2 = 4),$$

is true. Be aware that this is not completely consistent with the use of the word or in English. It often the case that or is exclusive, for example in the sentence:

You can have your cake or eat it.

However the most interesting way to combine two propositions is via implication: $p \implies q$, read p implies q, is true unless p is true and q is false.

In other words, if you start from a true statement, p, then it is okay to deduce a true statement, q, but it is not okay to deduce a false statement. So

$$(1 + 1 = 2) \implies (2 + 2 = 4)$$

is true but

$$(1 + 1 = 2) \implies (2 + 2 = 5)$$

is false.

In words, starting from the truth, one can only deduce other truths.

One curious thing about all of this is to wonder what happens if p is false. Then $p \implies q$ is always true! If q is true this doesn't seem so bad, but if q is false then at first this doesn't seem right. But if you start to write down examples, you begin to see it is okay

$$(1 = 2) \implies (2 = 3)$$

is true. Indeed, add 1 to both sides! If it really is the case that $1 = 2$ then in fact $2 = 3$.

In words, if you start with a false assumption, then you can draw any conclusion, true or false. If you start with a preposterous assumption then you can reach a preposterous conclusion.

There is an amusing example of this. Bertrand Russell was hard at work (together with Whitehead) on his magnum opus Principia Mathematica. He went to dinner in college and sat next to someone in a completely different field. Bertrand Russell explained that in mathematics false is defined as

$$0 = 1$$

and that from this proposition one can deduce any statement. The person he was sitting next to challenged Russell to prove he was the pope.

Here is Russell's proof:

We start with the equation

$$0 = 1.$$

Add one to both sides

$$1 = 2.$$

On the other hand, the statement

I and the pope are two different people.

is correct. As $1 = 2$ the statement

I and the pope are one person.

is correct. Therefore

I am the pope.

In short

$$(0 = 1) \implies (\text{I am the pope})$$

is true. This is our first proof.