HOMEWORK 9, DUE TUESDAY JUNE 6TH

1. (a) Prove that if A and B have the same cardinality then $\wp(A)$ and $\wp(B)$ have the same cardinality.
(b) Prove that if A_1 and A_2 have the same cardinality and B_1 and B_2 have the same cardinality, and $A_1 \cap B_1 = A_2 \cap B_2 = \emptyset$ then $A_1 \cup B_1$ has the same cardinality as $A_2 \cup B_2$.
(c) Prove that if A_1 and A_2 have the same cardinality and B_1 and B_2 have the same cardinality then $A_1 \times B_1$ has the same cardinality as $A_2 \times B_2$.
2. Let $n \in P$ be a positive integer.
(a) Prove that there is a unique natural number m such that

$$2^m \leq n < 2^{m+1}.$$

(b) Prove that there are unique distinct natural numbers $m_1, m_2, \dots, m_k$ such that

$$n = 2^{m_1} + 2^{m_2} + \dots + 2^{m_k}.$$

(c) Let

$$X = \{\, A \in \wp(\mathbb{N}) \mid A \text{ is finite} \,\}.$$

Show that X is countable.
3. Let a and b be two integers and assume that $a \neq 0$. Show that there are unique integers q and r such that

$$b = qa + r,$$

where $0 \leq r < |a|$.

Challenge problems/Just for fun:

4. Let A and B be two finite sets. Let B^A denote the set of all functions from A to B. Suppose that $m = |A|$ and $n = |B|$. Let $S \subset B^A$ denote the subset of all surjective functions from A to B. Give a formula for the cardinality of S and prove it is correct.