HOMEWORK 8, DUE TUESDAY MAY 30TH

1. Prove the binomial theorem by induction on n: if n is a natural number and x and y are indeterminates then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{i+j=n} \binom{i+j}{i} x^i y^j.$$

2. Let A and B be two finite sets. Let B^A denote the set of all functions from A to B.
(a) Prove that

$$|B^A| = |B|^{|A|}$$

(b) Let $I \subset B^A$ denote the subset of all injective functions from A to B. Show that

$$|I| = n(n-1)\dots(n-m+1),$$

where $n = |B|$ and $m = |A|$.
3. (a) Show that any two open intervals $(a,b) \subset \mathbb{R}$ and $(c,d) \subset \mathbb{R}$, where $a < b$ and $c < d$ are real numbers, have the same cardinality.
(b) Show that $(0,1)$ has the same cardinality as $\mathbb{R}$ (*Hint: consider rational functions*). Just write down a function that is a bijection; there is no need to prove your function is a bijection.
(c) Show that any open interval $(a,b) \subset \mathbb{R}$ where $a < b$ are real numbers, has the same cardinality as the real numbers.
4. Are the following functions injective or surjective? Justify your answers.
(a)

$$f\colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \qquad \text{given by} \qquad f(a,b) = 3a - 2b.$$

(b) If $A \subset X$ are two sets, define

$$l\colon \wp(X) \longrightarrow \wp(X) \qquad \text{by the rule} \qquad l(B) = A \bigtriangleup B.$$

(c) If $Y \subset X$ is a non-empty subset, then define

$$r\colon \wp(X) \longrightarrow \wp(Y) \qquad \text{by the rule} \qquad r(B) = Y \cap B.$$

5. Let I be the set of integers from 1 to n.

$$X_E = \{\, A \in \wp(I) \,|\, |A| \text{ is even} \,\} \qquad \text{and} \qquad X_O = \{\, A \in \wp(I) \,|\, |A| \text{ is odd} \,\}$$

(a) Show that we may define maps

$$f_1\colon X_E \longrightarrow X_O \qquad \text{and} \qquad f_2\colon X_O \longrightarrow X_E$$

by the common rule

$$A \longrightarrow A \triangle \{1\}.$$

(b) Show that X_E and X_O have the same cardinality.

Challenge problems/Just for fun:

6. Show that

(a)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(b)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

7. Let X be any set. Show that X and 2^X never have the same cardinality.