HOMEWORK 7, DUE TUESDAY MAY 23RD

1. Let X be a finite set, and let A, B and $A_1, A_2, \dots, A_n$ be subsets of X. Let $A^c = X \setminus A$ denote the complement.
(a) What is

$$\sum_{x \in X} \chi_A(x)?$$

(b) Use part (a) and the formulas you have proved about characteristic functions to conclude that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(c) Prove that

$$(A \cup B)^c = A^c \cap B^c.$$

(d) Prove that

$$\chi_{(\cup_{i=1}^n A_i)^c} = \prod_{i=1}^n (1 - \chi_{A_i}).$$

Here

$$\bigcup_{i=1}^n A_i$$

is the union of the sets $A_1, A_2, \dots, A_n$ and

$$\prod_{i=1}^n a_i$$

denotes the product of the numbers $a_1, a_2, \dots, a_n$.
(e) Prove that

$$\begin{aligned}\chi_{(\cup_{i=1}^n A_i)^c} &= \sum_{k=0}^n (-1)^k \sum_{i_1 < i_2 < i_3 < \cdots < i_k} \chi_{\cap_{j=1}^k A_{i_j}} \\ &= 1 - (\chi_{A_1} + \cdots + \chi_{A_n}) + (\chi_{A_1 \cap A_2} + \cdots + \chi_{A_{n-1} \cap A_n}) + \cdots + (-1)^n \chi_{A_1 \cap A_2 \cdots \cap A_n}.\end{aligned}$$

Here the second sum runs over all k tuples of distinct integers from 1 to n.
(f) Conclude that

$$|\bigcup_{i=1}^n A_i| = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \le i_1 < i_2 < i_3 < \cdots < i_k \le n} |\bigcap_{j=1}^k A_{i_j}|.$$

2. (a) Write out the formula from 1 (f) in the case when $k = 3$ and $A = A_1$, $B = A_2$ and $C = A_3$.
(b) How many integers between 1 and 1000 are not divisible by at least one of 2, 3 or 5? You may use the fact that the integer n is divisible by
(i) 2 and 3 if and only if it is divisible by 6,
(ii) 2 and 5 if and only if it is divisible by 10,
(iii) 3 and 5 if and only if it is divisible by 15,
(iv) 2, 3 and 5 if and only if it is divisible by 30.
3. Let $f\colon A \longrightarrow B$ be a function. Prove that
(a) f is injective if and only if either A is the emptyset or there is a function $g\colon B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A\colon A \longrightarrow A$.
(b) f is surjective if and only if there is a function $g\colon B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B\colon B \longrightarrow B$.
4. Suppose that $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ are two functions such that $g \circ f\colon A \longrightarrow C$ is a bijection.
Show that f is surjective if and only if g is injective.
Challenge problems/Just for fun:
5. (a) Let $G = (V, E)$ be a graph. Show that the degree function $d\colon V \longrightarrow \mathbb{N}$ is injective if and only if the number n of vertices is at most one.
(b) Classify all graphs G such that d misses at most one integer between 0 and $n - 1$.