HOMEWORK 6, DUE TUESDAY MAY 16TH

1. Prove or disprove:
(a)
$$\forall x \in \mathbb{R}, \ ((\forall \epsilon > 0, \ |x| < \epsilon) \implies x = 0)\,.$$
(b)
$$\forall x \in \mathbb{R}, \ \forall \epsilon > 0, \ (|x| < \epsilon \implies x = 0).$$
2. Let X be a set. If $A \subset X$ is a subset of X the **characteristic function** of A is the function
$$\chi_A \colon X \longrightarrow \{\,0,1\,\} \qquad \text{given by} \qquad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$
Let A and B be two subsets of X. Prove that
(a)
$$\chi_{A \cap B} = \chi_A \chi_B.$$
(this means that if $x \in X$ then
$$\chi_{A\cap B}(x) = \chi_A(x)\chi_B(x).$$
)
(b)
$$\chi_A + \chi_{X \setminus A} = \chi_X.$$
(c)
$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_A\chi_B.$$
(d)
$$\chi_{A \setminus B} = \chi_A - \chi_A\chi_B.$$
(e)
$$\chi_{A \triangle B} = \chi_A + \chi_B - 2\chi_A\chi_B.$$
Conclude that
$$(\chi_{A\triangle B}(x) = 1) \iff (\chi_A(x) + \chi_B(x) \text{ is odd}).$$
(f)
$$(\forall x \in X, \ \chi_A(x) \leq \chi_B(x)) \iff (A \subset B).$$
Conclude that
$$(\chi_A = \chi_B) \iff (A = B).$$
3. Let X be a set. Let
$$\{\,0,1\,\}^X,$$
denote the set of all functions from X to $\{\,0,1\,\}$.

Define a function

$$\Theta\colon \wp(X) \longrightarrow \{\,0,1\,\}^X,$$

by the rule

$$\Theta(A) = \chi_A.$$

Prove that Θ is a bijection.

4. Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be two functions and let $g \circ f\colon A \longrightarrow C$ be the composition. Prove that

(a) if f and g are injective then $g \circ f$ is injective.

(b) if f and g are surjective then $g \circ f$ is surjective.

(c) if f and g are bijective then $g \circ f$ is bijective.

Challenge problems/Just for fun:

5. Suppose you are a matchmaker tasked with the problem of marrying off a bunch of boys to a bunch of girls, with the proviso that everyone you marry off should already know each other. Show that you can always do this, if and only if, every subset of boys knows at least as many girls.