FINAL EXAM
MATH 109, UCSD, SPRING 17

You have three hours.

There are 10 problems, and the total number of points is 145. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________

Signature:______________________

Student ID #:______________________

Section instructor:______________________

Section Time:______________________

Problem	Points	Score
1	30	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
8	20	
9	20	
10	10	
11	10	
12	10	
13	10	
14	10	
Total	145	

1. (30pts) (i) *Give the definition of the absolute value of a real number.*

If r is a real number and r is non-negative then the absolute value of r is r; if r is negative the absolute value is $-r$.

(ii) *Give the definition of the difference of two sets.*

If A and B are two sets the difference $A \setminus B$ is the set of elements of A which are not elements of B.

(iii) *Give the definition of the symmetric difference of two sets.*

If A and B are two sets then the symmetric difference is the union of $A \setminus B$ and $B \setminus A$.

(iv) *Give the definition of a surjective function.*

A function $f\colon A \longrightarrow B$ is surjective if for every $b \in B$ we may find $a \in A$ such that $f(a) = b$.

(v) Give the definition of a bijective function.

$f\colon A \longrightarrow B$ is a bijective function if f is injective and surjective.

(vi) Give the definition of an invertible function.

$f\colon A \longrightarrow B$ is an invertible function if there is a function $g\colon B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B\colon B \longrightarrow B$ and $g \circ f = \mathrm{id}_A\colon A \longrightarrow A$.

2. (10pts) (i) *Prove that $|r|^2 = r^2$ for all real numbers r.*

There are two cases. If $r \geq 0$ then $|r| = r$ and so

$$|r|^2 = r^2.$$

If $r < 0$ then $|r| = -r$ and so

$$\begin{aligned} |r|^2 &= (-r)^2 \\ &= r^2. \end{aligned}$$

Either way, $|r|^2 = r^2$.

(ii) *Prove that if n is an integer then $n(n+1)$ is even.*

There are two cases. If n is even then $n = 2k$. In this case

$$n(n+1) = 2k(n+1)$$

is even as $k(n+1)$ is an integer.
Now suppose that n is odd. Let

$$A = \{\, a \,|\, a \text{ is even and } a \leq n \,\}.$$

Let m be the largest element of A and let $r = n - m$. Note that $m = 2k$, for some k. $r \geq 0$ as $m \leq n$. If $r = 0$ then $m = n$ is odd, a contradiction. If $r \geq 2$ then $m + 2 = 2k + 2 = 2(k+1)$ is even and $m + 2 \leq m + r = n$ so that $m < m + 2 \in A$. This contradicts the fact that m is the largest element of A. It follows that $r = 1$.
Thus $n = 2k + 1$. In this case

$$\begin{aligned} n(n+1) &= n(2k+2) \\ &= 2n(k+1), \end{aligned}$$

is even, as $n(k+1)$ is an integer.
Either way, $n(n+1)$ is even.

3. (10pts) *Let x and y be positive real numbers. Show that*

$$\sqrt{xy} \geq \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$

We have

$$(x-y)^2 = |x-y|^2 \geq 0.$$

Thus

$$x^2 - 2xy + y^2 \geq 0.$$

Adding $4xy$ to both sides we get

$$(x+y)^2 = x^2 + 2xy + y^2 \geq 4xy.$$

As x and y are positive real numbers, we have $x + y \geq 0$. Thus taking the square root of both sides

$$x + y \geq 2\sqrt{xy}.$$

Multiplying both sides by $\sqrt{xy} > 0$ we have

$$(x+y)\sqrt{xy} \geq 2xy.$$

Dividing both sides by $x + y > 0$ we have

$$\sqrt{xy} \geq \frac{2xy}{x+y} = \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$

4. (10pts) *Prove that*

$$n^3 \leq 2^n,$$

for all integers $n \geq 10$.

Let $P(n)$ be the statement that

$$n^3 \leq 2^n.$$

We prove that $P(n)$ holds for all natural numbers n at least ten.
If $n = 10$ then the LHS is

$$n^3 = 10^3 = 2^3 \cdot 5^3$$

and the RHS is

$$2^{10} = 2^3 \cdot 2^7.$$

Now $5^3 = 125$ and $2^7 = 128$, so that the

$$2^3 \cdot 5^3 \leq 2^3 \cdot 2^7.$$

Thus $P(10)$ holds.
Suppose that $P(k)$ holds and $k \geq 9$. We check that $P(k+1)$ holds.
We have

$$\begin{aligned}
(k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\
&\leq k^3 + 3k^2 + 3k^2 + 3k^2 \\
&= k^3 + 9k^2 \\
&\leq k^3 + k^3 \\
&\leq 2^k + 2^k \\
&= 2 \cdot 2^k \\
&= 2^{k+1},
\end{aligned}$$

where we used the fact that $P(k)$ holds to get from line four to line five. Thus $P(k+1)$ holds.
As we have checked that $P(10)$ holds and $P(k) \implies P(k+1)$, mathematical induction implies that $P(n)$ holds for all $n \geq 10$, that is

$$n^3 \leq 2^n$$

for all integers $n \geq 10$.

5. (10pts) *If*

$$A \subset \{ 1, 2, 3, \dots, n \}$$

then prove that

$$|A| \text{ is even} \qquad \text{if and only if} \qquad |A \triangle \{ 1 \}| \text{ is odd.}$$

First observe that n is even if and only if $n + 1$ is odd.
Suppose that $|A| = m$.
There are two cases. If $1 \notin A$ then

$$\begin{aligned} A \triangle \{1\} &= A \setminus \{1\} \cup \{1\} \setminus A \\ &= A \cup \{1\}. \end{aligned}$$

In this case

$$\begin{aligned} |A \triangle \{1\}| &= |A \cup \{1\} \\ &= |A| + 1 \\ &= m + 1. \end{aligned}$$

Thus $|A|$ is even if and only if $|A \triangle \{1\}|$ is odd
If $1 \in A$ then

$$\begin{aligned} A \triangle \{1\} &= A \setminus \{1\} \cup \{1\} \setminus A \\ &= A \setminus \{1\}. \end{aligned}$$

In this case

$$\begin{aligned} |A \triangle \{1\}| &= |A \setminus \{1\} \\ &= |A| - 1 \\ &= m - 1. \end{aligned}$$

As $m - 1$ is even if and only if m is odd, it follows that $m - 1$ is odd if and only if m is even. Thus $|A|$ is even if and only if $|A \triangle \{1\}|$ is odd
Either way, $|A|$ is even if and only if $|A \triangle \{1\}|$ is odd.

6. (10pts) *Prove or disprove:*

$$\forall \epsilon > 0,\ \exists N \in \mathbb{Z},\ (n \geq N) \implies \frac{1000}{n} < \epsilon.$$

This is true and so we prove it. Pick an integer N such that

$$N > \frac{1000}{\epsilon}.$$

If $n > N$ then

$$\begin{aligned}
\frac{1000}{n} &= \frac{1000}{n} \cdot 1 \\
&= \frac{1000}{n} \cdot \frac{\epsilon}{\epsilon} \\
&= \frac{\epsilon}{n} \cdot \frac{1000}{\epsilon} \\
&< \epsilon \cdot \frac{N}{n} \\
&< \epsilon \cdot \frac{n}{n} \\
&= \epsilon \cdot 1 \\
&= \epsilon.
\end{aligned}$$

7. (15pts) *Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be two functions.*
(a) Show that if f and g are injective then $g \circ f$ is injective.

Suppose that $(g \circ f)(a_1) = g \circ f(a_2)$. Let $b_i = f(a_1)$. We have

$$\begin{aligned} g(b_1) &= g(f(a_1)) \\ &= (g \circ f)(a_1) \\ &= (g \circ f)(a_2) \\ &= g(f(a_2)) \\ &= g(b_2). \end{aligned}$$

As g is injective, it follows that $b_1 = b_2$. Therefore

$$\begin{aligned} f(a_1) &= b_1 \\ &= b_2 \\ &= f(a_2). \end{aligned}$$

As f is injective, it follows that $a_1 = a_2$. Therefore $g \circ f$ is injective.

(b) Show that if f and g are surjective then $g \circ f$ is surjective.

Suppose that $c \in C$. As g is surjective, we may find $b \in B$ such that $g(b) = c$. As f is surjective, we may find $a \in A$ such that $f(a) = b$. We have

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) \\ &= g(b) \\ &= c. \end{aligned}$$

Thus $g \circ f$ is surjective.

(c) Show that if f and g are bijective then $g \circ f$ is bijective.

As f and g, f and g are injective and surjective. By (a), $g \circ f$ is injective and by (b) $g \circ f$ is surjective. But then $g \circ f$ is bijective.

8. (20pts) *(a) If A, B and C are three sets then find a formula* for $|A \cup B \cup C|$ and prove you formula is correct.

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C|.$$

We prove this by inclusion-exclusion. Suppose that $x \in A \cup B \cup C$. We consider how many times we count x, depending on where it lies. If x belongs to A but not B or C then x gets counted once as an element of A and it is not counted as any element of any other set on the RHS. If x belongs to $B \cap C$ but not A, it gets included once as an element of B, once as an element of C and excluded once as an element of $B \cap C$. If x belongs to $A \cap B \cap C$, it gets included three times, as elements of A, B and C, it gets excluded three times, as elements of $A \cap B$, $B \cap C$ and $A \cap C$ and it is included once as an element of $A \cap B \cap C$.
In all three cases, x is counted once in total. By symmetry x is always counted once. Thus the formula is correct.

(b) How many numbers between 1 and $10,000$ are not divisible by at least one of 2, 3 and 5 (you may use the standard properties of primes)?

Let A be the integers between 1 and $10,000$ divisible by 2, B be the integers between 1 and $10,000$ divisible by 3 and C be the integers between 1 and $10,000$ divisible by 5, so that

$$\begin{aligned} A &= \{\, k \in \mathbb{Z} \,|\, 1 \leq k \leq 10,000,\ k \text{ is divisible by } 2 \,\} \\ B &= \{\, k \in \mathbb{Z} \,|\, 1 \leq k \leq 10,000,\ k \text{ is divisible by } 3 \,\} \\ C &= \{\, k \in \mathbb{Z} \,|\, 1 \leq k \leq 10,000,\ k \text{ is divisible by } 5 \,\}. \end{aligned}$$

We use the formula in (a) to count the number of elements of $A \cup B \cup C$. These are the integers divisible by at least one of 2, 3, or 5. Suppose that $a \in A$. Then we can find $k \in \mathbb{Z}$ such that $a = 2k$. As $1 \leq a \leq 10,000$, we have $1 \leq k \leq 5000$. Thus

$$|A| = 5000$$

Similarly,

$$|C| = 2000.$$

Now $b \in B$ if and only if $b = 3k$, some integer k. As $1 \leq b \leq 10,000$, $1 \leq k \leq 3333$. Thus

$$|B| = 3333.$$

Now

$$A \cap B = \{\, k \in \mathbb{Z} \,|\, 1 \leq k \leq 10,000,\ k \text{ is divisible by } 6 \,\}.$$

Thus $a \in A \cap B$ if and only if $a = 6k$ for some integer k. We have $1 \leq k \leq 1666$. Thus

$$|A \cap B| = 1666.$$

Similarly

$$|A \cap C| = 10,000 \qquad \text{and} \qquad |B \cap C| = 666.$$

Finally,

$$A \cap B \cap C = \{\, k \in \mathbb{Z} \,|\, 1 \leq k \leq 10,000,\ k \text{ is divisible by } 30 \,\}.$$

Thus $a \in A \cap B \cap C$ if and only if $a = 30k$ for some integer k. We have $1 \leq k \leq 333$. Thus

$$|A \cap B \cap C| = 333.$$

It follows that

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C| \\ &= 5000 + 3333 + 2000 - 1666 - 1000 - 666 + 333 \\ &= 7334. \end{aligned}$$

Thus the number of integers between 1 and $10,000$ not divisible by one of 2, 3 or 5 is

$$7334.$$

9. (20pts) *(a) Let k and n be natural numbers and suppose that $k \leq n$. Prove that*

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We have

$$\begin{aligned}
\binom{n}{k+1} + \binom{n}{k} &= \frac{n!}{(k+1)!(n-k-1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k-1)!}\left(\frac{1}{k+1} + \frac{1}{n-k}\right) \\
&= \frac{n!}{k!(n-k-1)!}\left(\frac{(n-k)+(k+1)}{(k+1)(n-k)}\right) \\
&= \frac{n!}{k!(n-k-1)!}\left(\frac{(n+1)}{(k+1)(n-k)}\right) \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \binom{n+1}{k+1}.
\end{aligned}$$

(b) If n is a natural number and x and y are indeterminates then prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{i+j=n} \binom{i+j}{i} x^i y^j.$$

Let $P(n)$ be the statement that we have equality above. We prove $P(n)$ holds for all natural numbers n by induction on n. If $n = 0$ the LHS is

$$(x+y)^0 = 1,$$

and the RHS is

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} &= \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k} \\
&= \binom{0}{0} x^0 y^0 \\
&= 1.
\end{aligned}$$

As we have equality, $P(0)$ holds.
Now suppose that $P(m)$ holds. We check that $P(m+1)$ holds. We have

$$\begin{aligned}
(x+y)^{m+1} &= (x+y)(x+y)^m \\
&= (x+y)\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \\
&= x\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} + y\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \\
&= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} \\
&= \sum_{k=1}^{m+1} \binom{m}{k-1} x^k y^{m-k+1} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} \\
&= y^{m+1} + \sum_{k=1}^{m} \left(\binom{m}{k-1} + \binom{m}{k} \right) x^k y^{m-k+1} + x^{m+1} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m-k+1},
\end{aligned}$$

where we use the inductive hypothesis to get from line one to line two and (a) to get from line seven to line eight. Thus $P(k+1)$ holds.

Thus $P(n)$ holds for all natural numbers n, by mathematical induction, that is

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

10. (10pts) *Prove that if A is a set then $|A| < 2^{|A|}$.*

The function

$$g\colon A \longrightarrow \wp(A),$$

which sends an element a of A to the singleton set which contains a,

$$g(a) = \{ a \}$$

is easily seen to be injective.
Thus is the result does not hold then there is a surjection $f\colon A \longrightarrow \wp(A)$. We will derive a contradiction.
Let

$$B = \{ a \in A \mid a \notin f(a) \}.$$

Then $B \subset A$ so that $B \in \wp(A)$. As f is surjective, it follows that we may find $b \in A$ such that $f(b) = B$.
There are two cases. First suppose that $b \in B$. Then $b \in f(b)$ so that $b \notin B$, by definition of B. This is a contradiction.
Otherwise $b \notin B$. Then $b \notin f(b)$ so that $b \in B$, by definition of B. This is a contradiction.
Either way we get a contradiction and so there is no surjective function. Thus the cardinality of the powerset is greater than the cardinality of A.

Bonus Challenge Problems

11. (10pts) *Give a different proof of 9 (a).*

We count the number of ways to pick $k+1$ objects from $n+1$ objects.
Imagine one of the objects is red and the others n objects are blue.
If we pick $k+1$ then either we pick the red object or we don't.
If we do pick it then we have to pick k objects from the remaining n objects. There are

$$\binom{n}{k}$$

ways to do this.
If we don't pick the red object then we have to pick $k+1$ objects from the remaining n objects. There are

$$\binom{n}{k+1}$$

ways to do this.
Putting all of this together we get the formula:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

12. (10pts) *Show that for all non-negative integers m and n we have*

$$F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}.$$

where F_n is the Fibonacci sequence, $0, 1, 1, 2, 3, 5, 8, \ldots$.

See the lecture notes.

13. (10pts) *Prove that $\mathbb{R}$ is uncountable.*

Suppose not, suppose that the real numbers are countable. Then there would be a surjective function $f\colon \mathbb{N} \longrightarrow (0,1)$. Then we get a list of all real numbers between 0 and 1, $r_0, r_1, r_2, \dots$, where $r_i = f(i)$. Imagine making an actual list of these numbers

$$\begin{aligned} r_0 &= 0.a_{01}a_{02}a_{03}\dots \\ r_1 &= 0.a_{11}a_{12}a_{13}\dots \\ \vdots &= \vdots \\ r_n &= 0.a_{n1}a_{n2}a_{n3}\dots. \end{aligned}$$

We construct another real number r,

$$r = 0.a_1a_2a_3\dots,$$

as follows.

If the first digit a_{11} of r_1 is not one then we let the first digit a_1 of r be one. If the first digit a_{11} of r_1 is one then we let the first digit a_1 of r_1 be two.

If the second digit a_{22} of r_2 is not one then we let the second digit a_2 of r be one. If the first digit a_{22} of r_2 is one then we let the first digit a_2 of r_2 be two.

In general, we define the nth digit a_n of r as follows:

$$a_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1. \end{cases}$$

As f is surjective there is a natural number n such that $f(n) = r$, that is, $r = r_n$. Suppose that $n = 0$. There are two cases. Either $m = 0$ in which case by definition of m, $m_0 \neq 0$. Or $m = 1$ in which case $m_0 = 0$. Either way, $m \neq m_0$, which contradicts the fact that $r = r_0$. Thus $n \neq 1$.

Now suppose that $n > 0$. What is the nth digit a_n of r_n? If the nth digit is 1 then the nth digit of r_n is not equal to one. If the nth digit is 2 then the nth digit of r_n is equal to one. Either way, $a_n \neq a_{nn}$. This contradicts the fact that $r = r_n$.

Thus r does not belong to the list of real numbers. This contradicts the fact that f is surjective. Therefore the reals are uncountable.

14. (10pts) *Prove that if* $f\colon A \longrightarrow B$ *and* $g\colon B \longrightarrow A$ *are injective then* $|A| = |B|$.

See the lecture notes.