12. Noetherian Rings and Modules

First we need some more notation. We want to talk about both rings and modules being finitely generated. Therefore we need to introduce some notation for the subring generated by a set, distinguishable from the module (i.e ideal) generated by the same set

Definition 12.1. *Let R be a ring, let S be a subring and let X be a subset of R. The smallest ring generated by $X \cup S$ is denoted $S[X]$.*

We say that X generates R, over S, if $S[X] = R$.

We have already seen some examples of this. For example, the Gaussian integers are denoted $\mathbb{Z}[i]$. One can think of this as the smallest ring in $\mathbb{C}$ containing $\mathbb{Z}$ and i.

As already observed, if R is a ring, then R need not even be finitely generated as a ring, but as an R-module, R is generated by 1, $R = \langle 1 \rangle$.

Definition 12.2. *Let M be an R-module.*

We say that M is ***Noetherian*** *if every submodule is finitely generated. We say that a ring is* ***Noetherian*** *if it is Noetherian as a module over itself.*

Clearly every PID is Noetherian since, in a PID, every ideal has one generator. In particular every field is Noetherian and moreover every Euclidean domain is Noetherian, so that the polynomial ring over a field is Noetherian, and both $\mathbb{Z}$ and $\mathbb{Z}[i]$ are Noetherian.

Note that a vector space is Noetherian if and only if it has finite dimension.

Lemma 12.3. *An R-module M is Noetherian if and only if the set of submodules of M satisfies the ACC.*

Proof. Suppose that M is Noetherian and let

$$N_1 \subset N_2 \subset N_3 \subset \dots$$

be an ascending chain of submodules of M. Let N be the union. It is easy to check that N is a submodule of M. As M is Noetherian, N is finitely generated. Suppose that $n_1, n_2, \dots, n_k$ are generators of N. For each $1 \leq \alpha \leq k$, there is an index i_α, such that $n_\alpha \in N_{i_\alpha}$. Let i be the maximum of the i_α. Then $n_\alpha \in N_i$, for all $1 \leq \alpha \leq k$. It follows that

$$N = \langle n_1, n_2 \dots, n_k \rangle \subset N_i \subset N_l \subset N,$$

for all $l \geq i$. But then we must have equality, so that $N = N_l$, for all $l \geq i$. Thus the set of submodules of M satisfies the ACC.

Now suppose that the set of submodules of M satisfies the ACC. Suppose that M is not Noetherian. Let N be a submodule of M which is not finitely generated. Pick a sequence of elements of N, recursively, as follows. Put $n_0 = 0$. If we have already picked $n_1, n_2, \dots, n_k$, then let N_k be the submodule they generate. Clearly $N_k \subset N$. N_k must be a proper subset of N, as we are assuming that N is not finitely generated. Pick $n_{k+1} \in N - N_k$. Then

$$N_1 \subset N_2 \subset N_3 \subset \dots$$

is a strictly ascending chain of submodules, a contradiction. □

Definition 12.4. *Suppose that we have a sequence of R-modules,*

$$\dots M_{i-1} \xrightarrow{f} M_i \xrightarrow{g} M_{i+1} \dots$$

*We say that this sequence is **exact at** M_i, if the kernel of g is equal to the image of f. We say that this sequence is **exact**, if it is exact at each term. A **short exact sequence** is an exact sequence*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$$

Lemma 12.5.

(1)

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact at M if and only if f is injective.

(2)

$$M \xrightarrow{f} N \longrightarrow 0$$

is exact at N if and only if f is surjective.

Proof. Easy. □

Note that there is an obvious short exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0.$$

Note also the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

where the first map is multiplication by 2 (note that this map is a module homomorphism but not a ring homomorphism).

Lemma 12.6. *Let*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

be a short exact sequence of R-modules.

Then N is Noetherian if and only if M and P are Noetherian.

Proof. Suppose that N is Noetherian. Then M is (isomorphic to) a submodule of N. Thus any submodule of M is automatically a submodule of N and hence automatically finitely generated. Thus M is certainly Noetherian. There are two ways to see that P is Noetherian.

First suppose we are given a submodule P' of P. Let N' be the inverse image of P'. Then N' is a submodule of N. As N is Noetherian, it follows that N' has a finite number of generators. The image of these generators in P, generate the image of N', which as $N \longrightarrow P$ is surjective, is equal to P'. Thus P' is finitely generated.

Here is another way to prove that P is Noetherian. Pick an ascending chain of submodules

$$P_1 \subset P_2 \subset P_3 \subset \dots$$

of P. Let

$$N_1 \subset N_2 \subset N_3 \subset \dots$$

be the corresponding ascending chain in N. As N is Noetherian this chain must stabilise. Thus there is an index k, such that $N_i = N_j$, for all i and j at least k. As $N \longrightarrow P$ is surjective, it follows that the image of $N_i = P_i$. In particular $P_i = P_j$ for all i and j at least k. Thus every ascending chain of modules in P stabilises and P is Noetherian.

Now suppose that M and P are Noetherian. Suppose that

$$N_1 \subset N_2 \subset N_3 \subset \dots$$

is an ascending chain in N. Let P_i be the image of N_i and let M_i be the intersection of N_i with M. Then we get two ascending chains, one in M,

$$M_1 \subset M_2 \subset M_3 \subset \dots,$$

and one in P,

$$P_1 \subset P_2 \subset P_3 \subset \dots.$$

As M and P are Noetherian, both chains eventually stabilise. It follows that we can find a common index k, so that $M_i = M_j$ and $P_i = P_j$, for all i and j at least k. On the other hand, note that if $M_i = M_j$ and $P_i = P_j$, then in fact $N_i = N_j$. Thus every chain of submodules in N eventually stabilises. $\square$

Proposition 12.7. *Every finitely generated module M over a Noetherian ring R is Noetherian.*

Proof. By induction on n and the short exact sequence

$$0 \longrightarrow R^{n-1} \longrightarrow R^n \longrightarrow R \longrightarrow 0$$

we see that the direct sum of R with itself n times is Noetherian. On the other hand we have already seen that every finitely generated module is a quotient of a R^n, for some n. Thus M is a quotient of a Noetherian module and so it is Noetherian. $\square$

Theorem 12.8 (Hilbert's Basis Theorem)**.** *Let R be a Noetherian ring. Then $R[x]$ is Noetherian.*

Proof. Let I be an ideal of $R[x]$. We have to find a finite set of generators of I.

Let $J \subset R$ be the set of all coefficients of the leading terms of every element of I. I claim that J is an ideal in R. Suppose that a and b belong to J. Then there are two polynomials $f(x)$ and $g(x) \in I$ such that the leading coefficient of $f(x)$ is a and the leading coefficient of $g(x)$ is b,

$$f(x) = ax^m + \dots \qquad \text{and} \qquad g(x) = bx^n + \dots.$$

We may as well assume that $m \leq n$. Multiplying f by x^{n-m}, we may then assume that $m = n$. In this case the leading coefficient of $f + g$ is $a + b$. Now suppose that $r \in R$. Then $rf \in I$ has leading coefficient ra, so that $ra \in J$. Thus J is an ideal.

As R is Noetherian, J has a finite set of generators, $a_1, a_2, \dots, a_k$. Pick $f_i(x) \in I$ such that the leading coefficient of $f_i(x)$ is a_i. Let m be the maximum of the degrees d_i of each $f_i(x)$.

Pick a polynomial $f(x)$ in I. Suppose that the degree d of $f(x)$ is at least m. Let a be the leading coefficient of $f(x)$. As $a \in J$, we may find $r_1, r_2, \dots, r_k$ such that

$$a = r_1a_1 + r_2a_2 + \dots + r_ka_k.$$

Consider

$$g(x) = f(x) - \sum r_i x^{d-d_i} f_i(x).$$

Then the coefficient of x^d in $g(x)$ is zero, so that $g(x)$ has degree less than d. Continuing in this way, we can find $h(x) \in \langle f_1, f_2, \dots, f_k \rangle$, so that

$$f(x) = g(x) + h(x),$$

where $g(x)$ has degree at most $m-1$. Let M be the R-module generated by $1, x, x^2, \dots x^{m-1}$. Then $g(x) \in I \cap M$. As R is Noetherian and M is finitely generated, M is Noetherian. As $I \cap M$ is a submodule of M, it follows that we may find a finite set of generators $g_1, g_2, \dots, g_l$ for $I \cap M$. In this case $f_1, f_2, \dots, f_k, g_1, g_2, \dots, g_l$ are a finite set of generators for I.

Thus $R[x]$ is Noetherian. $\square$