HOMEWORK 9, DUE TUESDAY MARCH 8TH

The first few results refer to the power series ring which is defined as follows. Let R be a commutative ring and let x be an indeterminate. The power series ring in R, denoted $R[\![x]\!]$, consists of all (possibly infinite) formal sums,

$$\sum_{n\geq 0} a_n x^n,$$

where $a_n \in R$. Thus if $R = \mathbb{Q}$, then both

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots,$$

and

$$1 + 2!x + 3!x^2 + 4!x^3 + \dots,$$

are elements of $\mathbb{Q}[\![x]\!]$, even though the second, considered as a power series in the sense of analysis, does not converge for any $x \neq 0$. Addition and multiplication of elements of $R[\![x]\!]$ are defined as for polynomials. The degree of a power series is equal to the **smallest** n, so that the coefficient of a_n is non-zero. Even for a polynomial, in what follows the degree always refers to the degree as a power series.

1. (i) Show that $R[\![x]\!]$ is a ring.
(ii) Show that $f(x) \in R[\![x]\!]$ is a unit if and only if the degree of $f(x)$ is zero and the constant term is a unit. What is the inverse of $1 - x$?
(iii) Show that if R is an integral domain then the degree of a product is the sum of the degrees.
(iv) Show that if R is an integral domain then so is $R[\![x]\!]$.
(v) If F is a field then prove that $F[\![x]\!]$ is a Euclidean domain.
(vi) Show that if F is a field then $F[\![x]\!]$ is a UFD.
2. (i) See bonus problems.
(ii) Prove that if R is Noetherian then so is $R[\![x_1, x_2, \dots, x_n]\!]$, where the last term is defined appropriately.
3. Let M be a Noetherian R-module. If $\phi\colon M \longrightarrow M$ is a surjective R-linear map, prove that ϕ is an automorphism. (*Hint, consider the submodules,* $\operatorname{Ker}(\phi^n)$).
4. Let M, N and P be R-modules and let F be a free R-module of rank n. Show that there are isomorphisms, which are all natural (except for the last):

(a)

$$M \underset{R}{\otimes} N \simeq N \underset{R}{\otimes} M.$$

(b)

$$M \underset{R}{\otimes} (N \underset{R}{\otimes} P) \simeq (M \underset{R}{\otimes} N) \underset{R}{\otimes} P.$$

(c)

$$R \underset{R}{\otimes} M \simeq M.$$

(d)

$$M \underset{R}{\otimes} (N \oplus P) \simeq (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P).$$

(e)

$$F \underset{R}{\otimes} M \simeq M^n,$$

the direct sum of copies of M with itself n times.

5. Let m and n be integers. Identify $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_m$.

6. Show that if M and N are two finitely generated (respectively Noetherian) R-modules (respectively and R is Noetherian) then so is $M \underset{R}{\otimes} N$.

Bonus Problems 7. Show that if R is Noetherian then so is $R[\![x]\!]$.