HOMEWORK 5, DUE TUESDAY FEBRUARY 9TH

1. Let R be an integral domain. Let a and b be two elements of R. Show that if d and d' are both a gcd for the pair a and b, then d and d' are associates.
2. Let R be a UFD.
(a) Prove that for every pair of elements a and b of R, we may find an element $m = [a, b]$ that is a **least common multiple**, that is

(1) $a|m$ and $b|m$,
(2) and if $a|m'$ and $b|m'$ then $m|m'$.

Show that any two lcm's are associates.
(b) Show that if (a, b) denotes the gcd then $(a, b)[a, b]$ is an associate of ab.
Let R be a commutative ring. Our aim is to prove a very strong form of the Chinese Remainder Theorem. First we need some definitions. Let I and J be two ideals. The **sum** of I and J, denoted $I + J$, is the set consisting of all sums $i + j$, where $i \in I$ and $j \in J$. We say that I and J are **coprime** if $I + J = R$.
3. (a) Show that $I + J$ is an ideal of R.
(b) Show that I and J are coprime if and only if there is an $i \in I$ and a $j \in J$ such that $i + j = 1$.
(c) Show that if I and J are coprime then $IJ = I \cap J$.
Suppose that $I_1, I_2, \dots, I_k$ are ideals of R. We say these ideals are **pairwise coprime**, if for all $i \neq j$, I_i and I_j are coprime.
4. If $I_1, I_2, \dots, I_k$ are pairwise coprime, show that the product I of the ideals $I_1, I_2, \dots, I_k$ is equal to the intersection, that is

$$\prod_{i=1}^{k} I_i = \bigcap_{i=1}^{k} I_i.$$

(*Hint. Proceed by induction on* k).
Let R_i denote the quotient R/I_i. Define a map,

$$\phi \colon R \longrightarrow \oplus_{i=1}^{k} R_i,$$

by $\phi(a) = (a + I_1, a + I_2, \dots, a + I_k)$
4. (a) Show that ϕ is a ring homomorphism.
(b) See bonus problems.
(c) Show that ϕ is injective if and only if I, the product of the ideals $I_1, I_2, \dots, I_k$, is equal to the zero ideal.

5. Deduce the Chinese Remainder Theorem, which states that if $I_1, I_2, \ldots, I_k$ are pairwise coprime and the product I is the zero ideal, then R is isomorphic to $\oplus_{i=1}^{k} R_i$. Show how to deduce the other versions of the Chinese Remainder Theorem, which are stated as exercises in the book.
Bonus Problems 4 (b) Show that ϕ is surjective if and only if the ideals $I_1, I_2, \ldots, I_k$ are pairwise coprime.