HOMEWORK 4, DUE TUESDAY FEBRUARY 2ND

1. Let R be a ring and let I be an ideal of R, not equal to the whole of R. Suppose that every element not in I is a unit. Prove that I is the unique maximal ideal in R.
2. Let $\phi\colon R \longrightarrow S$ be a ring homomorphism and suppose that J is a prime ideal of S.
(i) Prove that $I = \phi^{-1}(J)$ is a prime ideal of R.
(ii) Give an example of an ideal J that is maximal such that I is not maximal.
3. Let R be an integral domain and let a and b be two elements of R. Prove that
(a) Show that $a|b$ if and only if $\langle b \rangle \subset \langle a \rangle$.
(b) a and b are associates if and only if $\langle a \rangle = \langle b \rangle$.
(c) Show that a is a unit if and only if $\langle a \rangle = R$.
4. Prove that every prime element of an integral domain is irreducible.
5. (a) Show that the elements 2, 3 and $1 \pm \sqrt{-5}$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.
(b) Show that every element of R can be factored into irreducibles.
(c) Show that R is not a UFD.
Bonus Problems 6. Let S be a commutative monoid, that is, a set together with a binary operation that is associative, commutative, and for which there is an identity, but not necessarily inverses. Treating this operation like multiplication in a ring, define what it means for S to have unique factorisation.
7. Let $v_1, v_2, \dots, v_n$ be a sequence of elements of $\mathbb{Z}^2$. Let S be the semigroup that consists of all linear combinations of $v_1, v_2, \dots, v_n$, with positive integral coefficients. Let the binary rule be ordinary addition. Determine which semigroups have unique factorisation.
8. Show that there is a ring R, such that every element of the ring is a product of irreducibles, whilst at the same time the factorisation algorithm can fail.