MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. §26: 1. Let

$$\phi\colon \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}\times\mathbb{Z}$$

be a ring homomorphism. Let $(a,b)=\phi(1,0)$ and let $(c,d)=\phi(0,1)$. Note that

$$(m,n) = m\cdot(1,0) + n\cdot(0,1)$$

and so

$$\begin{aligned}\phi(m,n) &= \phi(m\cdot(1,0)+n\cdot(0,1))\\ &= m\cdot\phi(1,0)+n\cdot\phi(0,1)\\ &= m\cdot(a,b)+n\cdot(c,d)\\ &= (ma+nc, mb+nd).\end{aligned}$$

It follows that if ϕ is a ring homomorphism, we just need to know where ϕ sends $(1,0)$ and $(0,1)$. Therefore it is enough to determine all possible choices for a, b, c and d.

We have

$$\begin{aligned}(a,b) &= \phi(1,0)\\ &= \phi((1,0)(1,0))\\ &= \phi(1,0)\phi(1,0)\\ &= (a,b)(a,b)\\ &= (a^2,b^2).\end{aligned}$$

Thus $a^2=a$ and $b^2=b$. Hence a and b belong to $\{0,1\}$.

By symmetry c and d also belong to $\{0,1\}$. We also have

$$\begin{aligned}(0,0) &= \phi(0,0)\\ &= \phi((1,0)(0,1))\\ &= \phi(1,0)\phi(0,1)\\ &= (a,b)(c,d)\\ &= (ac,bd).\end{aligned}$$

Thus $ac=0$ and $bd=0$. We write down all possible choices of a, b, c and d belonging to $\{0,1\}$ such that $ac=0$ and $bd=0$.

(1) All four of a, b, c and d are zero.
(2) One of a, b, c and d is one and the rest are zero.
(3) Two of a, b, c and d are one and the other two are zero.

(a) $a = b = 1$ and $c = d = 0$ or $a = b = 0$ and $c = d = 1$.
(b) $a = 1$, $b = 0$, $c = 0$, $d = 1$ or $a = 0$, $b = 1$, $c = 1$, $d = 0$.

We now check that in all of these cases we do indeed get a ring homomorphism.

In case (1) ϕ is the zero map, which is a ring homomorphism. In case (2) if $a = 1$ we get the map

$$(m, n) \longrightarrow (m, 0),$$

which is a ring homomorphism (it is the composition of projection onto the first factor and the inclusion $m \longrightarrow (m, 0)$). The other three cases are ring homomorphisms by symmetry.

In case 3 (a), $a = b = 1$ and $c = d = 0$, we get the map

$$(m, n) \longrightarrow (m, m),$$

which is a ring homomorphism (it is the composition of projection onto the first factor and the inclusion $m \longrightarrow (m, m)$). The other case is a ring homomorphism by symmetry.

In case 3 (b), $a = 1$, $b = 0$, $c = 0$, $d = 1$, we get the map

$$(m, n) \longrightarrow (m, n),$$

which is the identity. This is always a ring homomorphism.

In case 3 (b), $a = 0$, $b = 1$, $c = 1$, $d = 0$, we get the map

$$(m, n) \longrightarrow (n, m),$$

which it is easy to check is a ring homomorphism.

4. The elements of $2\mathbb{Z}/8\mathbb{Z}$ are the left cosets,

$$8\mathbb{Z}, \qquad 2 + 8\mathbb{Z}, \qquad 4 + 8\mathbb{Z}, \qquad \text{and} \qquad 6 + 8\mathbb{Z}.$$

The addition table is

$+$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$2+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$
$4+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$
$6+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$

and the multiplication table is

$*$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$
$2+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$
$4+8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$
$6+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$

This ring is not isomorphic to $\mathbb{Z}_4$. For example, the ring $\mathbb{Z}_4$ has unity and $2\mathbb{Z}/8\mathbb{Z}$ does not.

10. T: (a), (c), (e), (g), (h), (i), (j).
F: (b), (d), (f).
12. Let $R = \mathbb{Z}$ and let $I = 2\mathbb{Z}$. The quotient ring is isomorphic to $\mathbb{Z}_2$ which is a field.
18. Let $\phi\colon F \longrightarrow R$ be a ring homomorphism from a field to a ring. Let $I = \operatorname{Ker}\phi$. Then I is an ideal in F. If $I = \{0\}$ then ϕ is one to one.
Otherwise I contains a non-zero element a. As F is a field, a is a unit, and so we may find b such that $ab = 1$. But then $1 = ab \in I$ as $a \in I$ and I is an ideal. Now suppose that c is any element of F. Then $c = c1 \in I$ as $1 \in I$ and I is an ideal. In this case $I = F$. But then ϕ sends everything to zero and so it is the zero map.
20. We first check that ϕ is a group homomorphism. If a and $b \in R$ then

$$\begin{aligned}\phi(a+b) &= (a+b)^p \\ &= a^p + \binom{p}{1}a^{p-1}b + \binom{p}{2}a^{p-2}b^2 + \dots \binom{p}{i}a^i b^{n-1} + \dots + b^p \\ &= a^p + b^p \\ &= \phi(a) + \phi(b),\end{aligned}$$

where we used the fact that $\binom{p}{i}$ is zero for $0 < i < p$, as it is a multiple of p and the characteristic is p. Thus ϕ is a group homomorphism.
On the other hand

$$\begin{aligned}\phi(ab) &= (ab)^p \\ &= a^p b^p \\ &= \phi(a)\phi(b),\end{aligned}$$

so that ϕ is a ring homomorphism.
2. **Challenge Problems** §26:
30. We first check that I is an additive subgroup. $0 \in I$ as $0^1 = 0$. If a and $b \in I$ then $a^m = b^n = 0$ for some m and n. Consider $(a+b)^{m+n-1}$. If we use the binomial theorem to expand this we get terms of the form $a^i b^j$ where $i + j = m + n - 1$. If $i \geq m$ then

$$a^i b^j = a^m a^{i-m} b^j = 0.$$

If $i < m$ then $j = m + n - 1 - i \geq n$ and so

$$a^i b^j = a^i b^{j-n} b^n = 0.$$

Thus $(a+b)^{m+n-1} = 0$ and so $a+b \in I$. Thus I is closed under addition and so I is an additive subgroup.

If $a \in I$ and $r \in R$ then $a^n = 0$ for some n. But then

$$(ra)^n = r^n a^n = r^n 0 = 0,$$

so that $ra \in I$. It follows that I is an ideal.

31. If $a \in \mathbb{Z}_{12}$ is nilpotent then a^n is divisible by 12 for some n. This happens if a is divisible by both 2 and 3. Thus the nilradical of $\mathbb{Z}_{12}$ is $\{0, 6\}$.

$\mathbb{Z}$ is an integral domain, so the nilradical is the zero ideal $\{0\}$.

If $a \in \mathbb{Z}_{32}$ is nilpotent then a^n is divisible by 32 for some n. This happens if a is even. The nilradical is the set of even elements of $\mathbb{Z}_{32}$.