MODEL ANSWERS TO THE SECOND HOMEWORK

§13:
2. ϕ is not a homomorphism.

$$\phi(0.5+0.5)=\phi(1)=1 \qquad \text{but} \qquad \phi(0.5)+\phi(0.5)=0+0=0\neq 1,$$

so that

$$\phi(0.5+0.5)\neq\phi(0.5)+\phi(0.5).$$

3. ϕ is a homomorphism. Suppose that x and y are real numbers.

$$\begin{aligned}\phi(xy)&=|xy|\\&=|x||y|\\&=\phi(x)\phi(y).\end{aligned}$$

4. ϕ is a homomorphism. Suppose that a and $b\in\mathbb{Z}_6$. We have to check that

$$\phi(a+b)=\phi(a)+\phi(b).$$

There are three cases:

- a and b are both even.
- one of a and b is even.
- neither a nor b is even.

If a and b are both even then the $a+b$ is even and both sides are 0.
If one of a and b is odd and the other is even then $a+b$ is odd and both sides are $1=1+0$.
If both a and b are odd then $a+b$ is even and both sides are $0=1+1$.
Thus ϕ is a homomorphism.
6. ϕ is a homomorphism. Suppose that x and y are real numbers.

$$\begin{aligned}\phi(x+y)&=2^{x+y}\\&=2^x2^y\\&=\phi(x)\phi(y).\end{aligned}$$

8. ϕ is not a homomorphism. The problem is when G is not abelian. For example, consider $G=S_3$, the simplest non-abelian group. Take $a=(1,2)$ and $b=(2,3)$. Then

$$\phi((1,2)(2,3))=\phi(3,1,2)=\phi(1,2,3)=(1,2,3)^{-1}=(1,3,2).$$

On the other hand,

$$\phi(1,2)\phi(2,3)=(1,2)(2,3)=(3,1,2)=(1,2,3)\neq(1,3,2).$$

Therefore

$$\phi((1,2)(2,3)) \neq \phi(1,2)\phi(2,3).$$

16. The identity in $\mathbb{Z}_2$ is zero. The elements of S_3 which are sent to zero are the even permutations. So the kernel of ϕ is

$$\operatorname{Ker}\phi = A_3 = \{\, e, (1,2,3), (1,3,2) \,\}.$$

19. Let $\sigma = \phi(1)$. As we have a homomorphism

$$\phi(2) = \sigma^2, \qquad \phi(3) = \sigma^3 \qquad \text{and more generally} \qquad \phi(n) = \sigma^n.$$

We compute

$$\sigma = (1,4,2,6)(2,5,7) = (2,5,7,6,1,4).$$

Since we get a 6-cycle, σ has order 6.
Therefore the kernel of ϕ is all multiples of 6,

$$\operatorname{Ker}\phi = 6\mathbb{Z}.$$

$$\phi(20) = \sigma^{20} = \sigma^{18}\sigma^2 = \sigma^2 = (2,7,1)(5,6,4).$$

32. T: (a), (b), (d), (g), (h) (unless *into* means one to one)
F: (c), (e), (f), (i), (j).
35. Let $\pi\colon \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ denote projection onto the first factor. Then π is a homomorphism. If $\psi\colon \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_5$ denote the natural injection then ψ is a homomorphism. Finally the composition

$$\phi = \psi \circ \pi\colon \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_5$$

is a non-trivial homomorphism, as the composition of group homomorphisms is a homomorphism.
44. Let H be the kernel of ϕ. $\phi[G]$ is in one to one correspondence with the left cosets of H in G. So $|\phi[G]|$ is equal to the number of left cosets. By Lagrange this is finite and divides the order of G.
50. Suppose that $\phi[G]$ is abelian. If x and $y \in G$ then

$$\begin{aligned} \phi(xy) &= \phi(x)\phi(y) \\ &= \phi(y)\phi(x) \\ &= \phi(yx). \end{aligned}$$

Multiplying on the right by $\phi(yx)^{-1}$ we get

$$\begin{aligned} e' &= \phi(xy)\phi(yx)^{-1} \\ &= \phi(xy)\phi((yx)^{-1}) \\ &= \phi(xy(yx)^{-1}) \\ &= \phi(xyx^{-1}y^{-1}). \end{aligned}$$

Therefore $xyx^{-1}y^{-1} \in \operatorname{Ker}\phi$.
Now suppose that $xyx^{-1}y^{-1} \in \operatorname{Ker}\phi$.

$$\begin{aligned} e' &= \phi(xyx^{-1}y^{-1}) \\ &= \phi(xy(yx)^{-1}) \\ &= \phi(xy)\phi((yx)^{-1}) \\ &= \phi(xy)\phi(yx)^{-1}. \end{aligned}$$

Multiplying on the right by $\phi(yx)$, it follows that $\phi(xy) = \phi(yx)$. But then

$$\begin{aligned} \phi(x)\phi(y) &= \phi(xy) \\ &= \phi(yx) \\ &= \phi(y)\phi(x). \end{aligned}$$

It follows that $\phi[G]$ is abelian.