MODEL ANSWERS TO THE FIRST HOMEWORK

§10

5.

$\{0, 18\}$, $1+\{0, 18\} = \{1, 19\}$, $2+\{0, 18\} = \{2, 20\}, \dots$, $17+\{0, 18\} = \{17, 35\}$.

6. The group D_4 has order eight and the subgroup $H = \{\rho_0, \mu_2\}$ has order two and so the number of cosets is 4. One coset is H. Pick an element not in H, for example, ρ_1,

$$\rho_1 H = \{\rho_1, \delta_2\}.$$

Pick an element not in either of these two left cosets, for example, ρ_2,

$$\rho_2 H = \{\rho_2, \mu_1\}.$$

This leaves two elements, which must form their own coset,

$$\rho_3 H = \{\rho_3, \delta_1\}.$$

15. We first multiply out σ to represent it as a product of disjoint cycles,

$$(1, 2, 4, 5)(2, 3) = (2, 3, 4, 5, 1) = (1, 2, 3, 4, 5).$$

So σ is a 5-cycle and the order of σ is five. The order of S_5 is $5! = 5 \cdot 4! = 120$. So the index of σ is $4! = 24$.

19. T: (a), (b), (c), (e), (g), (h), (j).

F: (d), (f), (i) (The Klein 4-group has no element of order 4).

27. Define

$$\phi \colon H \longrightarrow Hg \qquad \text{by the rule} \qquad h \longrightarrow hg.$$

Suppose that $y \in Hg$. Then $y = hg$ for some h and $\phi(h) = hg = y$. Thus ϕ is onto. Suppose that $\phi(h_1) = \phi(h_2)$. Then $h_1 g = h_2 g$. Multiplying both sides by g^{-1} on the right, we get $h_1 = h_2$. But then ϕ is one to one.

30. False. Take $G = S_3$ and $H = \{e, (1, 2)\}$. Let $a = (1, 3, 2)$ and $b = (2, 3)$. Then $a \in aH$ and

$$a = (1, 3, 2) = (2, 3)(1, 2) \in bH,$$

so that $aH = bH$. But

$$Hb = \{(2, 3), (1, 2, 3)\},$$

so that $a \notin Hb$. As $a \in Ha$, $Ha \neq Hb$.

§11

2. The elements of $\mathbb{Z}_3 \times \mathbb{Z}_4$ are $(0,0)$, $(1,0)$, $(2,0)$, $(0,1)$, $(1,1)$, $(2,1)$, $(0,2)$, $(1,2)$, $(2,2)$, $(0,3)$, $(1,3)$, $(2,3)$. The order of an element is the lcm of the orders of the components:
1: $(0,0)$
2: $(0,2)$
3: $(1,0)$, $(2,0)$
4: $(0,1)$, $(0,3)$
6: $(1,2)$, $(2,2)$
12: $(1,1)$, $(2,1)$, $(1,3)$, $(2,3)$.
Yes, this group is cyclic. For example, $(1,1)$ is a generator.
7. The order of 3 in $\mathbb{Z}_4$ is 4; the order of 6 in $\mathbb{Z}_{12}$ is 2; the order of 12 in $\mathbb{Z}_{20}$ is 5; the order of 16 in $\mathbb{Z}_{24}$ is 3.
So the order of $(3,6,12,16)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$ is 60, the lcm of 4, 2, 5 and 3.
10. The order of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is 8. By Lagrange the order of a subgroup is 1, 2, 4, or 8. If the order is 1 the subgroup is the trivial subgroup and if the order is 8 we have all of G. So we list the subgroups of order 2 and 4. Every element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, other than the identity, has order two. Thus the subgroups of order two are:

$$\{ (0,0,0),(1,0,0) \} \quad \{ (0,0,0),(0,1,0) \} \quad \{ (0,0,0),(0,0,1) \} \quad \{ (0,0,0),(1,1,1) \}$$
$$\{ (0,0,0),(0,1,1) \} \quad \{ (0,0,0),(1,0,1) \} \quad \{ (0,0,0),(1,1,0) \}.$$

If you take any two elements of order two and add them together this gives three elements of order two and together with the identity this is a subgroup of order 4. Thus the subgroups of order four are:

$$\begin{array}{ll}
\{ (0,0,0),(0,1,0),(0,0,1),(0,1,1) \} & \{ (0,0,0),(1,0,0),(0,0,1),(1,0,1) \} \\
\{ (0,0,0),(1,0,0),(0,1,0),(1,1,0) \} & \{ (0,0,0),(0,1,1),(1,0,0),(1,1,1) \} \\
\{ (0,0,0),(1,0,1),(0,1,0),(1,1,1) \} & \{ (0,0,0),(1,1,0),(0,0,1),(1,1,1) \} \\
\{ (0,0,0),(0,1,1),(1,0,1),(1,1,0) \}. &
\end{array}$$

12. The Klein 4 group is the unique group of order 4 not isomorphic to a cyclic group. $\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 4 and it is not cyclic, so it is isomorphic to the Klein 4 group.
Every element of the Klein 4 group has order one or two. The elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ of order two are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times 2\mathbb{Z}_4$ and this group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus the subgroups isomorphic to the

Klein group are:

$$\{(0,0,0),(0,1,0),(0,0,2),(0,1,2)\} \qquad \{(0,0,0),(1,0,0),(0,0,2),(1,0,2)\}$$
$$\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\} \qquad \{(0,0,0),(0,1,2),(1,0,0),(1,1,2)\}$$
$$\{(0,0,0),(1,0,2),(0,1,0),(1,1,2)\} \qquad \{(0,0,0),(1,1,0),(0,0,2),(1,1,2)\}$$
$$\{(0,0,0),(0,1,2),(1,0,2),(1,1,0)\}.$$

16. Yes. Both groups are abelian of order $24 = 2^3 \cdot 3$. By the fundamental theorem of finitely generated abelian groups, there are three abelian groups of order 24 up to isomorphism:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \qquad \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \qquad \text{and} \qquad \mathbb{Z}_8 \times \mathbb{Z}_3.$$

Consider the elements of order a non-trivial power of 2. The first group has elements only of order 2, the second group has elements of order 2 and 4 and the third group has elements of order 2, 4 and 8.
The group $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ has elements of order four but not eight. Thus $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ is isomorphic to the second group in the list.
The group $\mathbb{Z}_4 \times \mathbb{Z}_6$ also has elements of order four but not eight. Thus $\mathbb{Z}_4 \times \mathbb{Z}_6$ is also isomorphic to the second group in the list.
But then $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ are isomorphic.
24. We first write down the prime factorisation of $720 = 72 \cdot 10 = 2^4 \cdot 3^2 \cdot 5$.
Using the fundamental theorem of finitely generated abelian groups the abelian groups of order 720, up to isomorphism are:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \quad \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5, \quad \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5, \quad \mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \quad \mathbb{Z}_{16} \times \mathbb{Z}_9 \times \mathbb{Z}_5.$$

47. H contains the identity by assumption. Suppose that $h \in H$. Then $h^2 = e$, the identity. Hence $h^{-1} = h \in H$ and so H is closed under taking inverses. Now suppose that $h_i \in H$, $i = 1$ and 2. Then

$$\begin{aligned}(h_1h_2)^2 &= h_1h_2h_1h_2\\ &= h_1^2h_2^2\\ &= e,\end{aligned}$$

where we got from the first line to the second line as G is abelian. Therefore either h_1h_2 is the identity or it is has order two. In particular $h_1h_2 \in H$ and H is closed under multiplication. Therefore H is a subgroup of G.

52. Suppose that G is a cyclic group. Then every subgroup H is cyclic. Every element of $H = \mathbb{Z}_p \times \mathbb{Z}_p$ has order either 1 or p and the order of H is p^2 and so H is not cyclic. Therefore H is not isomorphic to a subgroup of a cyclic group G.
Now suppose that G does not contain a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. The fundamental theorem of finitely generated abelian groups implies that G is isomorphic to

$$\mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}},$$

where $p_1, p_2, \dots, p_n$ are primes and $a_1, a_2, \dots, a_n$ are positive invegers. Suppose that $p_i = p_j$. Then G contains a subgroup isomorphic to $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$. As $\mathbb{Z}_a$ contains a subgroup isomorphic to $\mathbb{Z}_p$, $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, a contradiction.
Thus $p_i = p_j$ implies that $i = j$. But then G is a cyclic group.

Challenge Problems

45. We may assume that $G = \mathbb{Z}_n$. If d divides n then let $a = n/d$. Then

$$\langle a \rangle$$

is a subgroup of order d.
Now let H be a subgroup of G of order d. Then d divides n by Lagrange. On the other hand, the smallest element a of H is a generator of H. The order of a is n/a, so that $d = n/a$. But then $a = n/d$ and so there is only one subgroup of order d.

46. Partition the elements of $\mathbb{Z}_n$ by their order. By Lagrange the order must be a divisor d of n. Let P_d be the elements of order d. Then

$$n = |\mathbb{Z}_n| = \sum_{d|n} |P_d|.$$

Now every element of P_d generates a subgroup of order d. But there is only such subgroup H. H is isomorphic to $\mathbb{Z}_d$ and $a \in \mathbb{Z}_d$ generates $\mathbb{Z}_d$ if and only if a is coprime to d. Thus

$$|P_d| = \phi(d).$$

Putting all of this together, we have

$$n = \sum_{d|n} \phi(d).$$

47. Let n be the order of G. Partition the elements of G by their order. By Lagrange the order must be a divisor d of n. Let P_d be the elements of order d. Then

$$n = |G| = \sum_{d|n} |P_d|.$$

If $x^m = e$ always has at most m solutions then there is at most one subgroup of order m and so $|P_d| \leq \phi(d)$. Since we already saw that

$$n = \sum_{d|n} \phi(d).$$

we must have that $|P_d| = \phi(d)$ for all divisors d of n. In particular $|P_n| = \phi(n) \neq 0$ and so there are elements of order n. But then G is cyclic.