FIRST MIDTERM
MATH 103B, UCSD, SPRING 16

You have 50 minutes.

There are 6 problems, and the total number of points is 85. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Problem	Points	Score
1	15	
2	15	
3	10	
4	15	
5	15	
6	15	
7	10	
8	10	
Total	85	

1. (15pts) (i) *Give the definition of the index of a subgroup H of a group G.* The number of left cosets of H in G.

(ii) *Give the definition of the direct product $H \times G$ of two groups H and G.* The Cartesian product of H and G with multiplication defined by the rule

$$(h_1, g_1)(h_2, g_2) = (h_1 h_2, g_1 g_2),$$

for all $h_i \in H$ and $g_i \in G$, $i = 1, 2$.

(iii) *Give the definition of a group homomorphism $\phi\colon G \longrightarrow G'$.* A map between two groups such that

$$\phi(ab) = \phi(a)\phi(b),$$

for all a and $b \in G$.

2. (15pts) (i) *List the elements of* $\mathbb{Z}_2 \times \mathbb{Z}_5$.

$(0,0) \quad (0,1) \quad (0,2) \quad (0,3) \quad (0,4) \quad (1,0) \quad (1,1) \quad (1,2) \quad (1,3) \quad (1,4)$.

(ii) *Find the order of these elements.*
Order 1: $(0,0)$
Order 2: $(1,0)$
Order 5: $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$
Order 10: $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$

(iii) *Is this group cyclic?*
Yes. For example $(1,1)$ is a generator.

3. (10pts) *What is the order of* $(4, 3, 5, 12) \in \mathbb{Z}_{12} \times \mathbb{Z}_{18} \times \mathbb{Z}_{40} \times \mathbb{Z}_{120}$*?*
The order of an element of the product of groups is the lowest common multiple of the orders of the entries.
4 has order 3 in $\mathbb{Z}_{12}$; 3 has order 6 in $\mathbb{Z}_{18}$; 5 has order 8 in $\mathbb{Z}_{40}$; 12 has order 10 in $\mathbb{Z}_{120}$.
The lcm of 3, 6, 8 and 10 is $2^3 \cdot 3 \cdot 5 = 120$. The order is 120.

4. (15pts) (i) *State the fundamental theorem of finitely generated abelian groups.*
Every finitely generated abelian group is isomorphic to a product

$$\mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z},$$

where $p_1, p_2, \ldots, p_n$ are prime numbers and $a_1, a_2, \ldots, a_n$ are positive integers. The direct product is unique, up to re-ordering the factors, so that the number of copies of $\mathbb{Z}$ and the prime powers are unique.

(ii) *Find all abelian groups of order* 56*, up to isomorphism.*
We first write down the prime factorisation of $56 = 8 \cdot 7$. So the possibilities are:

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$.
(2) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_7$.
(3) $\mathbb{Z}_8 \times \mathbb{Z}_7$.

5. (15pts) Determine whether the given map ϕ is a group homomorphism.
(i) $\phi\colon \mathbb{R} \longrightarrow \mathbb{Z}$ *under addition, given by* $\phi(x)$ = *the greatest integer* $\leq x$.
ϕ is not a homomorphism.

$$\phi(0.5+0.5)=\phi(1)=1 \qquad \text{but} \qquad \phi(0.5)+\phi(0.5)=0+0=0\neq 1,$$

so that

$$\phi(0.5+0.5)\neq \phi(0.5)+\phi(0.5).$$

(ii) $\phi\colon\ \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$ *given by* $\phi(A)=\det A$, *where* $\mathrm{GL}(n,\mathbb{R})$ *is the group of all* $n\times n$ *invertible matrices with real entries.*
Suppose that A and $B\in \mathrm{GL}(n,\mathbb{R})$. We have

$$\begin{aligned}\phi(AB)&=\det(AB)\\&=\det A\det B\\&=\phi(A)\phi(B).\end{aligned}$$

Thus ϕ is a group homomorphism.

6. (15pts) *Let* $\phi\colon G \longrightarrow G'$ *be a group homomorphism. Suppose that* G *is abelian and* ϕ *is onto. Show that* G' *is abelian.*
Suppose that a' and b' are elements of G'. As ϕ is onto we may find elements a and b of G such that $\phi(a) = a'$ and $\phi(b) = b'$. We have

$$\begin{aligned} a'b' &= \phi(a)\phi(b) \\ &= \phi(ab) \\ &= \phi(ba) \\ &= \phi(b)\phi(a) \\ &= b'a'. \end{aligned}$$

Therefore G' is abelian.

Bonus Challenge Problems

7. (10pts) *Sketch a proof that every finite group of isometries of the plane is isomorphic to either* $\mathbb{Z}_n$ *or* D_n *for some* n*.*

See the lecture notes.

8. (10pts) *Let G be a finite abelian group of order n. Show that there is a subgroup H of order m if and only if m divides n.*
See the lecture notes.