2. Plane isometries

Definition 2.1. *We say that a permutation $\phi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an **isometry** if ϕ preserves distances, that is, the distance between two points P and Q is the same as the distance between their images $\phi(P)$ and $\phi(Q)$.*

Isometries are sometimes also called rigid motions.

Lemma 2.2. *The set of all plane isometries is a subgroup of the group of all permutations of $\mathbb{R}^2$.*

Proof. Suppose that ϕ and ψ are two isometries and let $\xi = \psi \circ \phi$ be the composition. Then

$$\xi(P) = \psi(\phi(P)) \qquad \text{and} \qquad \xi(Q) = \psi(\phi(Q)).$$

Then the distance between $\xi(P)$ and $\xi(Q)$ is the same as the distance between $\phi(P)$ and $\phi(Q)$, as ψ is an isometry. On the other hand, the distance between $\phi(P)$ and $\phi(Q)$ is the same as the distance between P and Q. Thus the distance between $\xi(P)$ and $\xi(Q)$ is the same as the distance between P and Q.

Thus ξ is an isometry and the set of all plane isometries is closed under composition.

The identity map is obviously an isometry. If ϕ is an isometry then so is ϕ^{-1}. Thus the set of all isometries contains the identity and is closed under taking inverses.

It follows that the set of all isometries is a subgroup of the permutation group. □

In fact isometries come in four different types:

- translation τ: Slide every point by the same vector, that is, by the same distance and the same direction.
- rotation ρ: Rotate every point around a fixed point P through an angle θ.
- reflection μ: Reflect every point across a line L.
- glide reflection γ: The composition of a translation and a reflection in a line fixed by the translation.

For example, $\gamma(x, y) = (x - 3, -y)$ is a glide reflection in the x-axis.

We can separate these four types into two pairs: the first two **preserve orientation** and the second two **reverse orientation**; if you take a clock and apply an orientation reversing isometry the clock will run backwards.

Given a subset S of $\mathbb{R}$ one can look at the subgroup of isometries which fix S (as a set).

Theorem 2.3. *Every finite group of isometries of the plane is isomorphic to either $\mathbb{Z}_n$ or to a dihedral group D_n, for some positive integer n.*

Sketch of proof. Suppose that $\phi_1, \phi_2, \dots, \phi_m$ are the elements of G. Let

$$P_i = (x_i, y_i) = \phi_i(0, 0)$$

and set

$$P = (\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + \cdots + x_m}{m}, \frac{y_1 + y_2 + \cdots + y_m}{m} \right).$$

Then P is the centroid of the points $P_1, P_2, \dots, P_m$. Suppose that $\phi_j \in G$. Then $\phi_j \phi_i = \phi_k \in G$ some k and so

$$\phi_j(P_i) = \phi_j(\phi_i(0, 0)) = \phi_k(0, 0) = P_k.$$

Thus the elements of G permute the points $P_1, P_2, \dots, P_m$ and so they fix the centroid P.

Looking at the four possible types of isometry only two of them fix a point, rotation and reflection. Consider the orientation preserving elements H of G. These are the rotations. A rotation only fixes one point, so the elements of H are rotations about the centroid. Since the product of two rotations about the same point is a rotation, H is a subgroup of G. Let θ be the smallest angle of rotation. It is not hard to see that every element represents a rotation through a multiple of θ. In other words, if ρ represents rotations about P through an angle of θ then

$$H = \langle \theta \rangle,$$

a cyclic subgroup of G. Note that the product of two orientation reversing isometries is orientation preserving. So either every element of G is orientation preserving or m is even and half the elements are orientation preserving. In the first case $G = H \simeq \mathbb{Z}_m$.

Otherwise G contains one reflection μ about a line L through P. In this case the coset $H\mu$ contains all of the reflections. Pick a point $Q \neq P$ on the line L and consider the regular n-gon given by the images of Q under rotation. Then the elements of H correspond to all rotations of the n-gon and μ corresponds to a reflection about all line through oppositve vertices of the n-gon. Thus G is isomorphic to the dihedral group D_n. □

It is interesting to think a little bit about infinite groups of symmetries. We start with symmetries of a *discrete frieze.* Start with a pattern of bounded width and height and repeat it along an infinite strip. This is the sort of pattern you might see along the wall of a room. The symmetries of such a pattern is called a **frieze group**.

For example, suppose we start with an integral sign translated by one unit horizontally in both directions. One obvious symmetry is translation by one unit τ. But we may pick the centre of any integral sign and rotate by 180°, call this ρ. One can check that

$$\rho^{-1}\tau\rho = \tau^{-1}.$$

If one compares this with what happens for the Dihedral group D_n, it is natural to call this infinite frieze group D_∞.

Another possibility is to replace the integral sign by a D. In this case as well as the translation τ one can reflect in a horizontal line; call this isometry μ. In this case the two isometries commute and the group of isometries is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. Yet another possibility is to replace D with A. In this case one can reflect in a vertical line and the resulting isometry group is again D_∞.

A much more sophisticated example arises if one takes a sequence of two rows of D's, where the top row is shifted halfway across. In this case there is a glide reflection; translate half way across and then flip along the horizontal line dividing the two rows.

In fact there is a complete classification of all possible groups which arise:

$$\mathbb{Z}, \qquad D_\infty, \qquad \mathbb{Z} \times \mathbb{Z}_2, \qquad D_\infty \times \mathbb{Z}_2.$$

Note that the same group is associated with different patterns.

It is also interesting to consider what happens if you tile the plane by translating a figure in two different directions; the resulting group of isometries is called a **wallpaper group** or a **crystallographic group**.

One possibility is to start with a unit square and translate it both horizontally and vertically one unit. The symmetry group of this pattern obviously contains $\mathbb{Z} \times \mathbb{Z}$, the translations in both directions. But it also contains the symmetries of a square D_4.