14. Field of fractions

If R is an integral domain we have seen that the cancellation laws hold,

$$\text{if } ab = ac \text{ and } a \neq 0 \qquad \text{then} \qquad b = c.$$

One obvious reason why the cancellation laws might hold is that a has a multiplicative inverse. Multiply both sides by a^{-1} and it is clear why we can cancel a.

The integers $\mathbb{Z}$ are an integral domain but not every element has an inverse. For example 2 does not have an inverse. On the other hand the integers sit naturally inside the rational numbers $\mathbb{Q}$. The rational numbers are a field so every non-zero element has an inverse.

In fact the rational numbers are constructed from the integers. We would like to mirror this construction for any integral domain. For every integral domain R we would like to construct a field F which contains R (or at least a copy of R).

The rational numbers $\mathbb{Q}$ are constructed from the integers $\mathbb{Z}$ by adding inverses. In fact a rational number is of the form a/b, where a and b are integers. (We will use the same notation for a field F, we will write $1/b$ for the multiplicative inverse of b and so we will write a/b for $ab^{-1} = b^{-1}a$). Note that a rational number does not have a unique representative in this way. In fact

$$\frac{a}{b} = \frac{ka}{kb},$$

where k is any integer. So really a rational number is an equivalence class of pairs $[a, b]$, where two such pairs $[a, b]$ and $[c, d]$ are equivalent if and only if $ad = bc$.

Now given an arbitrary integral domain R, we will perform the same construction.

Definition-Lemma 14.1. *Let R be any integral domain. Let N be the subset of $R \times R$ such that the second coordinate is non-zero.*

Define an equivalence relation $\sim$ on N as follows.

$$(a, b) \sim (c, d) \qquad \textit{if and only if} \qquad ad = bc.$$

Proof. We have to check three things, reflexivity, symmetry and transitivity.

Suppose that $(a, b) \in N$. Then

$$a \cdot b = a \cdot b$$

so that $(a, b) \sim (a, b)$. Hence $\sim$ is reflexive.

Now suppose that (a,b), $(c,d) \in N$ and that $(a,b) \sim (c,d)$. Then $ad = bc$. But then $cb = da$, as R is commutative and so $(c,d) \sim (a,b)$. Hence $\sim$ is symmetric.

Finally suppose that (a,b), (c,d) and $(e,f) \in R$ and that $(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$. Then $ad = bc$ and $cf = de$. Then

$$\begin{aligned}(af)d &= (ad)f \\ &= (bc)f \\ &= b(cf) \\ &= (be)d.\end{aligned}$$

As $(c,d) \in N$, we have $d \neq 0$. Cancelling d, we get $af = be$. Thus $(a,b) \sim (e,f)$. Hence $\sim$ is transitive. □

Definition-Theorem 14.2. *The* ***field of fractions of*** *R, denoted F, is the set of equivalence classes, under the equivalence relation defined above. Given two elements $[a,b]$ and $[c,d]$ define*

$$[a,b] + [c,d] = [ad + bc, bd] \qquad \text{and} \qquad [a,b] \cdot [c,d] = [ab, cd].$$

With these rules of addition and multiplication F becomes a field. Moreover there is a natural one to one ring homomorphism

$$\phi \colon R \longrightarrow F,$$

so that we may identify R as a subring of F.

Proof. First we have to check that this rule of addition and multiplication is well-defined. Suppose that $[a,b] = [a',b']$ and $[c,d] = [c',d']$. By commutativity and an obvious induction (involving at most two steps, the only real advantage of which is to simplify the notation) we may assume $c = c'$ and $d = d'$. As $[a,b] = [a',b']$ we have $ab' = a'b$. Thus

$$\begin{aligned}(a'd + b'c)(bd) &= a'bd^2 + bb'cd \\ &= ab'd^2 + bb'cd \\ &= (ad + bc)(b'd).\end{aligned}$$

Thus $[a'd + b'c, b'd] = [ad + bc, bd]$. Thus the given rule of addition is well-defined. It can be shown similarly (and in fact more easily) that the given rule for multiplication is also well-defined.

We leave it is an exercise for the reader to check that F is a commutative ring under addition and that multiplication is associative. For example, note that $[0,1]$ plays the role of 0 and $[1,1]$ plays the role of 1.

Given an element $[a,b]$ in F, where $a \neq 0$, then it is easy to see that $[b,a]$ is the inverse of $[a,b]$. It follows that F is a field.

Define a map

$$\phi \colon R \longrightarrow F,$$

by the rule

$$\phi(a) = [a, 1].$$

Again it is easy to check that ϕ is indeed a one to one ring homomorphism. $\square$

Note that the field of fractions F is the smallest field containing R.

Example 14.3. *If $R = \mathbb{Z}$ then the field of fractions is (isomorphic to) $\mathbb{Q}$.*

The integers are contained in many other fields. For example $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{Z} \subset \mathbb{C}$. Both fields contain the rational numbers.

It is in fact the case that the field of fractions is unique in a well-defined sense.

Theorem 14.4. *Let F be the field of quotients of an integral domain R and let L be any field containing R.*

Then there is a ring homomorphism $\psi \colon F \longrightarrow L$ which is an isomorphism of F with its image $\psi(F) \subset L$.

Example 14.5. *The Gaussian integers R are all complex numbers whose real and imaginary part are integers.*

$$R = \{\, a + bi \mid a, b \in \mathbb{Z} \,\} \subset \mathbb{C}.$$

It is easy to see that R is closed under addition and multiplication of complex numbers, so that R is a subring of the complex numbers. Therefore R is commutative, contains no zero-divisors and contains unity, $1 \neq 0$. Therefore R is an integral domain.

The complex numbers $\mathbb{C}$ are a field. Therefore the field of fractions of R sits naturally inside the complex numbers. The Gaussian integers contain the integers and so the field of fractions of the Gaussian integers must contain the rational numbers.

Hence the field of fractions of the Gaussian integers must contain all complex numbers whose real and imaginary part are rationals:

$$S = \{\, a + bi \mid a, b \in \mathbb{Q} \,\} \subset \mathbb{C}.$$

We check that all ratios of Gaussian integers are of this form:

$$\begin{aligned}
\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\
&= \frac{ac+bd+(bc-ad)i}{c^2+d^2} \\
&= \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2} \in S,
\end{aligned}$$

as

$$\frac{ac+bd}{c^2+d^2} \qquad \text{and} \qquad \frac{bc-ad}{c^2+d^2}$$

are rational numbers.