0. Conjugation equal translation

We introduce a way to think about conjugation which makes computing the conjugate of a permutation very easy.

Definition 0.1. *Let g and h be two elements of a group G.*
The element ghg^{-1} is called the ***conjugate*** *of h by g.*

One reason why conjugation is so important, is because it measures how far the group G is from being abelian.
Indeed if G were abelian, then
$$gh = hg.$$
Multiplying by g^{-1} on the right, we would have
$$h = ghg^{-1}.$$
Thus G is abelian if and only if the conjugate of every element by any other element is the same element.
Another reason why conjugation is so important, is that really conjugation is the same as translation.

Lemma 0.2. *Let σ and τ be two elements of S_n. Suppose that $\sigma = (a_1, a_2, \dots, a_k)(b_1, b_2, \dots, b_l)\dots$ is the cycle decomposition of σ.*
Then $(\tau(a_1), \tau(a_2), \dots, \tau(a_k))(\tau(b_1), \tau(b_2), \dots, \tau(b_l))\dots$ is the cycle decomposition of $\tau\sigma\tau^{-1}$, the conjugate of σ by τ.

Proof. Since both sides of the equation
$$\tau\sigma\tau^{-1} = (\tau(a_1), \tau(a_2), \dots, \tau(a_k))(\tau(b_1), \tau(b_2), \dots, \tau(b_l))\dots$$
are permutations, it suffices to check that both sides have the same effect on any integer j from 1 to n. As τ is onto, $j = \tau(i)$ for some i. By symmetry, we may as well assume that $j = \tau(a_1)$. Then $\sigma(a_1) = a_2$ and the right hand side maps $\tau(a_1)$ to $\tau(a_2)$. But
$$\begin{aligned}\tau\sigma\tau^{-1}(\tau(a_1)) &= \tau\sigma(a_1)\\ &= \tau(a_2).\end{aligned}$$
Thus the LHS and RHS have the same effect on j and so they must be equal. □

In other words, to find compute the conjugate of σ by τ, just translate the elements of the cycle decomposition of σ. For example suppose
$$\sigma = (3, 7, 4, 2)(1, 6, 5)$$
in S_8 and τ is
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 1 & 8 & 7 & 6 & 4 \end{pmatrix}.$$

Then the conjugate of σ by τ is

$$\tau\sigma\tau^{-1} = (5, 6, 1, 2)(3, 7, 8).$$