PRACTICE PROBLEMS FOR THE MIDTERM

Here are some practice problems for the first midterm culled from various locations (several problems are a bit more involved than the midterm problems but are hopefully useful for review):

1. Suppose that

$$u\colon \mathbb{C} - \{0\} \longrightarrow \mathbb{R},$$

is a harmonic function. Show that u is surjective.

2. Let $\mathbb{H}$

$$\{\, z \in \mathbb{C} \mid \operatorname{Im} z > 0 \,\}$$

be the upper half plane and let f denote the principal branch of the logarithm restricted to $\mathbb{H}$. Consider the sequence of iterates

$$f, \qquad f \circ f, \qquad f \circ f \circ f, \dots,$$

Is this sequence of functions locally bounded in $\mathbb{H}$? Explain.

3. State and prove Hadamard's three-circles theorem.

4. Let u be a bounded harmonic function in the first quadrant,

$$U = \{\, z \in \mathbb{C} \mid \operatorname{Re} z > 0, \operatorname{Im} z > 0 \,\}.$$

Suppose that

$$\lim_{z \to x} u(z) = 1,$$

for all $x \in (0, 1)$ and otherwise

$$\lim_{z \to z_0} u(z) = 0,$$

where z_0 is any other part of the boundary.
What is

$$u\left(\frac{1+i}{\sqrt{2}}\right)?$$

5. Using Cauchy's integral formula, write down the value of a holomorphic function $f(z)$ where $|z| < 1$ in terms of a contour integral around the unit circle, $\zeta = e^{i\theta}$.
By considering the point $1/\bar{z}$ show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} \, \mathrm{d}\theta.$$

By setting $z = re^{i\alpha}$, show that for any harmonic function $u(r, \alpha)$,

$$u(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) \frac{1 - r^2}{1 - 2r\cos(\alpha - \theta) + r^2} \, \mathrm{d}\theta.$$

Assuming that the harmonic conjugate $v(r, \theta)$ can be written as

$$v(r, \alpha) = v(0) + \frac{1}{\pi} \int_0^{2\pi} u(1, \theta) \frac{r \sin(\alpha - \theta)}{1 - 2r \cos(\alpha - \theta) + r^2} \, \mathrm{d}\theta,$$

deduce that

$$f(z) = iv(0) + \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) \frac{\zeta + z}{\zeta - z} \, \mathrm{d}\theta.$$

6. Show that if u is a harmonic function on the unit disc Δ continuous on the closure and u agrees with a real-valued polynomial on the boundary then u is a real-valued polynomial.
7. Give the definition of Euler's constant.
8. Give the definition of the Gamma function.