MODEL ANSWERS TO THE FOURTH HOMEWORK

1. Consider the family

$$\mathfrak{G} = \{\, g = e^{-f} \mid f \in \mathfrak{F} \,\}.$$

If $g \in \mathfrak{G}$ then $|g| \leq 1$. Then $\mathfrak{G}$ is uniformly bounded by 1. It follows that $\mathfrak{G}$ is a normal family in the usual sense. If $f_1, f_2, \dots$ is a sequence of elements of $\mathfrak{F}$ then let $g_i = e^{-f_i}$. We have that $g_1, g_2, \dots$ is a sequence of elements of $\mathfrak{G}$. Possibly passing to a subsequence we may assume that they converge to a holomorphic function g, uniformly on compact subsets.

As $g_1, g_2, \dots$ are nowhere zero, Hurwitz implies that the limit is either identically zero or it is nowhere zero. In the former case it follows that the real part of $f_1, f_2, \dots$ is going to infinity so that $f_1, f_2, \dots$ tend to infinity, uniformly on compact subsets.

Now suppose that g is nowhere zero. As U is the union of finitely many simply connected regions there is no harm in assuming that U is simply connected so that we may define a branch of the logarithm $\log z$ on U. Let $h_i(z) = \log g_i(z)$. The difference $h_i(z) - f_i(z)$ is both continuous and a multiple of $2\pi i$. Thus it is constant. Note that $h_i(z)$ tends to the holomorphic function $h(z) = \log g(z)$ uniformly on compact subsets.

There are two cases. The difference $h_i(z) - g_i(z)$ is bounded or not. If the difference is bounded then it takes on only finitely many values; possibly passing to a subsequence we may assume that it is constant. Rechoosing the branch of our logarithm we may assume that $f_i(z) = h_i(z)$ and $f_i(z)$ tends to the holomorphic function $f(z) = h(z)$ uniformly on compact subsets.

Now suppose that that the difference is unbounded. Possibly passing to a subsequence we may assume that the imaginary part of $f_i(z)$ tends to infinity, so that the difference of the imaginary part of $f_i(z)$ and $h(z)$ tends to infinity. But then $f_i(z)$ tends to infinity, uniformly on compact subsets, since the imaginary part of $h(z)$ is bounded on compact subsets.

2. If we work in the disc Δ then the sequence of functions $f_n(z) = z^n$ tends uniformly on compact subsets to 0; if we work in the complement of the closed unit disc $|z| > 1$ then the same sequence tends uniformly to ∞.

Now suppose that the region contains a point $|z_0| = 1$. Then the region contains a small ball about this point. If we take a point in this ball

such that $|z| > 1$ then $f_n(z)$ tends to infinity. But if we choose a point $|z| < 1$ then $f_n(z)$ tends to zero. Therefore this is not a normal family.
3. Suppose that $\mathfrak{F}$ is a normal family. If f is not a polynomial then f has an essential singularity at ∞. Pick $r_1 < s_1 < s_2 < r_2$, let $s = s_2/s_1$ and let $k_i = s^i$. Let $g(z) = k_i z$.
Then

$$E_n = \{ g(z) \in \mathbb{C} \mid s_1 \leq |z| \leq s_2 \} = \{ z \in \mathbb{C} \mid s^i s_1 \leq |z| \leq s^i s_2 \}.$$

The union of these annuli is the set

$$\{ z \in \mathbb{C} \mid |z| > s_1 \},$$

a neighbourhood of ∞.
By Casorati-Weierstrass $f_i(z) = f(k_i z)$ is neither bounded nor is the limit constant. Thus $\mathfrak{F}$ is not a normal family.
Now suppose that $f(z)$ is a polynomial. Suppose that we have a sequence of complex numbers $k_1, k_2, \ldots$. There are two cases. Suppose first that k_n is unbounded. Then there is a subsequence which converges to ∞. Replacing $k_1, k_2, \ldots$ by a subsequence we may suppose that $k_1, k_2, \ldots$ converges to infinity. If f is constant $f_n(z) = f(k_n z)$ is constant. Otherwise $f_n(z)$ tends uniformly to infinity on the annulus.
Now suppose that k_n is bounded. Then there is a convergent subsequence. Replacing $k_1, k_2, \ldots$ by a subsequence we may suppose that $k_1, k_2, \ldots$ converges to $k \in \mathbb{C}$. In this case $f_n(z) = f(k_n z)$ converges to $f(kz)$ uniformly on the annulus.
4. By assumption there is a sequence of elements $f_1, f_2, \ldots$ of $\mathfrak{F}$ and a compact subset E of U such that no subsequence converges uniformly on E. For each $k \in \mathbb{N}$ we can consider the subdivision of $\mathbb{C}$ into squares of width $1/k$ with sides parallel to the real and imaginary axes and one corner at the origin, so that the vertices of the squares are at the points $a/k + b/ki$, where a and b are integers. This induces a subdivision of E and the sequence cannot converge on one of the finitely many subsets E_k of this subdivision. In this way we construct a nested sequence of compact subsets on which $f_1, f_2, \ldots$ does not converge. By compactness we may find a point z_0 common to each E_k. Suppose that V is an open neighbourhood of z_0. Then $E_k \subset V$ for some k and so $f_1, f_2, \ldots$ is not a normal family when restricted to V.
5. Consider the map $g\colon U \longrightarrow \Delta$ given by $g(z) = \bar{f}(\bar{z})$. Then g is biholomorphic, $g(z_0) = 0$ and $g'(z_0) > 0$. Thus $g = f$ by uniqueness and so $\bar{f}(z) = \bar{\bar{f}}(\bar{z}) = f(\bar{z})$.