MODEL ANSWERS TO THE SECOND HOMEWORK

1. Possibly replacing ρ by a slightly smaller number we may assume that u is harmonic on the closed disc $|z| \leq \rho$. In particular we may assume that u is continuous on the closed disc $z| = \rho$. Let U be the restriction of u to the circle $|z| = \rho$. Let $v = P_U$ be the harmonic function given by the Poisson integral.
Pick $\epsilon > 0$ and let

$$w(z) = u(z) - v(z) + \epsilon \log r/\rho.$$

As $u(z)$ is bounded and $v(z)$ is harmonic on the whole disc, $w(z)$ tends to $-\infty$ as z tends to zero. Consider a circle of radius δ centred at the origin. Then w is a harmonic function on the annulus $\delta \leq |z| \leq \rho$. The maximum is achieved on the boundary. On the circle $|z| = \rho$, $w(z) = 0$ and if $\delta > 0$ is sufficiently small then $w(z) < 0$ on $|z| = \delta$. Thus $w(z) \leq 0$. Letting $\epsilon > 0$ we see that

$$u(z) \leq v(z).$$

Replacing u by $-u$ we get the reverse inequality. Thus $u(z) = v(z)$ and u extends to a harmonic function v.
2. Suppose that $f(z)$ is identically zero on the circle $|z| = r_1$. By the reflection principle one can extend $f(z)$ to a holomorphic function in a neighbourhood of any point where $|z| = r_1$ (just apply a Möbius transformation so that the circle $|z| = r_1$ is mapped to the real axis). But then $f(z)$ is identically zero and there is nothing to prove. Similarly if $f(z)$ is identically zero on $|z| = r_2$. Thus we may assume that $M(r_1)$ and $M(r_2)$ are positive.
Let

$$u(z) = a \log |z| + \log |f(z)|.$$

Then u is harmonic away from the zeroes of $f(z)$, as it is a linear combination of harmonic functions. Put small circles around the zeroes of $f(z)$. By the maximum principle the maximum of u occurs on the boundary, which is either on the two big circles $|z| = r_1$ and $|z| = r_2$ or on one of the small circles. But if the circles are small enough then $\log |f(z)|$ is large and negative so that the maximum is on one of the big circles.
Thus

$$a \log r + \log M(r) \leq \max(a \log r_1 + \log M(r_1), a \log r_2 + \log M(r_2)).$$

Now pick a so that

$$a \log r_i + \log M(r_i)$$

is independent of i. Then

$$a = \frac{\log M(r_2) - \log M(r_1)}{\log r_1 - \log r_2},$$

and so

$$\begin{aligned}
\log M(r) &\leq a(\log r_2 - \log r) + \log M(r_2) \\
&= \frac{(\log M(r_2) - \log M(r_1))(\log r_2 - \log r) + \log M(r_2)(\log r_1 - \log r_2)}{(\log r_1 - \log r_2)} \\
&= \frac{\log M(r_2)(\log r - \log r_1) + \log M(r_1))(\log r_2 - \log r)}{(\log r_2 - \log r_1)} \\
&= \log M(r_1)^{\alpha} M(r_2)^{1-\alpha}.
\end{aligned}$$

If we have equality then $u(z)$ must be constant. But then

$$|f(z)| = |z^b|,$$

for some constant b, so that

$$f(z) = e^{i\theta} z^b,$$

for some constant θ.

3. Let

$$f \colon \Delta \longrightarrow \mathbb{H},$$

be the Möbius transformation given by

$$w \longrightarrow z = i\frac{w+1}{1-w}.$$

Then f is biholomorphic, and sends the unit circle to the upper half-plane. The inverse transformation is

$$z \longrightarrow w = \frac{z-i}{z+i}.$$

Let $V(\theta) = U(f(\theta)) = U(\xi)$. Then V is piecewise continuous and

$$P_V(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\theta}-w|^2} V(\theta)\, d\theta,$$

is harmonic in the unit disc with boundary values $V(\theta)$ at points of continuity. Now make the substitution

$$e^{i\theta} = \frac{\xi - i}{\xi + i}.$$

Then

$$d\theta = \frac{2(\xi + i)\, d\xi}{(\xi + i)^2(\xi - i)} = \frac{2\, d\xi}{\xi^2 + 1},$$

and

$$\begin{aligned}
\frac{1 - |w|^2}{|e^{i\theta} - w|^2} &= |\xi + i|^2 \frac{|z + i|^2 - |z - i|^2}{|(\xi - i)(z + i) - (\xi + i)(z - i)|^2} \\
&= (\xi^2 + 1)\frac{x^2 + (y + 1)^2 - x^2 - (y - 1)^2}{4|\xi - z|^2} \\
&= (\xi^2 + 1)\frac{4y}{4(x - \xi)^2 + y^2} \\
&= (\xi^2 + 1)\frac{y}{(x - \xi)^2 + y^2},
\end{aligned}$$

which gives the result.

4. Let u be a harmonic function on the upper half plane, which is continuous on the real axis. Let U be the restriction of u to the real axis and let $P_U(z)$ be the Poisson integral. Then $u - P_U$ is harmonic and zero on the real axis. Now suppose that u is bounded. Pick $\epsilon > 0$ and consider

$$v(z) = u(z) - P_U(z) - \epsilon \operatorname{Im}(\sqrt{iz}).$$

Note that for $\operatorname{Im} z \geq 0$, the argument of iz lies between $\pi/2$ and $3\pi/2$ so that $\sqrt{iz}$ is a holomorphic function and $\operatorname{Im}(\sqrt{iz})$ is harmonic and moreover the argument of $\sqrt{iz}$ lies between $\pi/4$ and $3\pi/4$.

Thus v tends to zero to $-\infty$ as z tends to ∞. Consider the region from $-R$ to R along the real axis and a semicircle of radius R to R to $-R$. The maximum of $v(z)$ occurs on the boundary. If R is large enough the maximum is on the real axis and so the maximum is zero.

It follows that

$$u - P_U \leq \epsilon \operatorname{Im}(\sqrt{iz}) \leq 0.$$

Letting ϵ go to zero, we get

$$u \leq P_U.$$

Replacing u by $-u$ we get the reverse inequality

$$P_U \leq u.$$

But then $u = P_U$, as required.