9. The Riemann Zeta function II

Recall

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,\mathrm{d}x,$$

for $\sigma > 1$.

If we replace x by nx in the integral then we obtain

$$n^{-s}\Gamma(s) = \int_0^\infty x^{s-1} e^{-nx}\,\mathrm{d}x.$$

Now sum over n to get

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x.$$

Note that as $\sigma > 1$ the integral is absolutely convergent at both ends, $x = 0$ and $x = \infty$ and so we can switch the order of integration and summation. Also we define

$$x^{s-1} = e^{(s-1)\log x}$$

unambiguously, in the usual way.

Now we define two paths C and C_n. For C we come in from positive infinity just above the real axis, describe most of a small circle centred at the origin and return to infinity just below the real axis. We don't care too much about the exact definition of C except that the circle has radius r less than 2π. For C_n we start at point of C describe most of a square encompassing $\pm 2k\pi i$, $0 \leq k \leq n$ and end at point of C just below the x-axis. We then describe the bounded part of C to complete a full cycle.

Theorem 9.1. *If $\sigma > 1$ then*

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}\,\mathrm{d}z,$$

where $(-z)^{s-1}$ is defined on the complement of the positive real x-axis as

$$e^{(s-1)\log(-z)} \qquad \text{with} \qquad -\pi < \operatorname{Im}\log(-z) < \pi.$$

Proof. The integral obviously converges. By Cauchy's theorem the integral does not depend on C, as long as C does not go around any non-zero multiples of $2\pi i$. In particular we are free to let the radius of the circle go to zero.

Consider the integral around the circular part of C. As r goes to zero the length of the path is proportional to r. As the denominator is also proportional to r and $(-z)^{s-1}$ goes to zero the integral around the circular part goes to zero.

We are left with an integral along the positive real axis described both ways. On the upper edge

$$(-z)^{s-1} = x^{s-1}e^{-(s-1)\pi i},$$

and on the lower edge

$$(-z)^{s-1} = x^{s-1}e^{(s-1)\pi i}.$$

It follows that

$$\begin{aligned}\int_C \frac{(-z)^{s-1}}{e^z-1}\,\mathrm{d}z &= -\int_0^\infty \frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}\,\mathrm{d}x + \int_0^\infty \frac{x^{s-1}e^{(s-1)\pi i}}{e^x-1}\,\mathrm{d}x\\ &= 2i\sin\pi(s-1)\zeta(s)\Gamma(s).\end{aligned}$$

Now use the fact that

$$\sin\pi(s-1) = -\sin\pi s \qquad \text{and} \qquad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}. \qquad \square$$

Corollary 9.2. *The ζ-function can be extended to a meromorphic function on the whole complex plane whose only pole is a simple pole at $s=1$ with residue 1.*

Proof. Consider the RHS of the equation in (9.1). $\Gamma(1-s)$ is meromorphic on $\mathbb{C}$ and the integral defines an entire function. Since the RHS is meromorphic on the whole complex plane we can use this equation to extend $\zeta(s)$ to a meromorphic function on the whole complex plane.

$\Gamma(1-s)$ has poles at $s=1$, $s=2$, ... But we already know that $\zeta(s)$ is holomorphic for $\sigma > 1$ so the zeroes of the integral must cancel with the poles and the only pole is at the origin.

As $s=1$, $\Gamma(1-s)$ has a simple pole with residue 1. On the other hand

$$\int_C \frac{1}{e^z-1}\,\mathrm{d}z = 2\pi i,$$

by the Residue theorem. Thus $\zeta(s)$ has residue one at $s=1$. $\square$

We can calculate $\zeta(-n)$ where $n \in \mathbb{N}$ explicitly. We already know that

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_1^\infty (-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1}.$$

We have

$$\zeta(-n) = (-1)^n\frac{n!}{2\pi i}\int_C \frac{z)^{-n-1}}{e^z-1}\,\mathrm{d}z.$$

Thus $\zeta(-n)$ is $(-1)^n n!$ times the coefficient of z^n in the expansion above.

Thus

$$\zeta(0) = -1/2 \qquad \zeta(-2m) = 0 \qquad \text{and} \qquad \zeta(-2m+1) = \frac{(-1)^m B_m}{2m}.$$

The points $-2m$ are called the *trivial zeroes* of the ζ-function.

Note that if $\sigma > 1$ we have

$$|\zeta(s)| \leq \zeta(\sigma).$$

Thus we have good control on the ζ-function for $\sigma > 1$. In fact we also have good control for $\sigma < 0$:

Theorem 9.3.

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$

Proof. We use the path C_n. We assume that square part is defined by the lines $t = \pm(2n+1)\pi$ and $\sigma = \pm(2n+1)\pi$. The cycle $C_n - C$ has winding number one about the points $\pm 2m\pi i$ with $m = 1, 2, \dots, n$. The poles at these points of

$$\frac{(-z)^{s-1}}{e^z - 1}$$

are simple with residues

$$(\mp 2m\pi i)^{s-1}.$$

Thus

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1}\,\mathrm{d}z &= \sum_{m=1}^{n} \left[(-2m\pi i)^{s-1} + (2m\pi i)^{s-1}\right] \\
&= \sum_{m=1}^{n} (2m\pi)^{s-1}(i^{s-1} + (-i)^{s-1}) \\
&= 2\sum_{m=1}^{n} (2m\pi)^{s-1}(i^s - (-i)^s)/2i \\
&= 2\sum_{m=1}^{n} (2m\pi)^{s-1}\sin\frac{\pi s}{2},
\end{aligned}$$

where we used the fact that $i = e^{\pi i/2}$. We divide C_n into two parts, $C_n' + C_n''$, where C_n' is the square bit and C_n'' is the rest. It is easy to see that $|e^z - 1|$ is bounded below on C_n' by a fixed positive constant, independent of n, while $|(-z)^{s-1}|$ is bounded by a multiple of n^{s-1}. The length of C_n' is of the order of n and so

$$\left|\int_{C_n'} \frac{(-z)^{s-1}}{e^z - 1}\,\mathrm{d}z\right| \leq A n^{\sigma},$$

for some constant A. If $\sigma < 0$ then the integral over C'_n will tend to zero as n tends to infinity and the same is true for the integral over C''_n. Therefore the integral over $C_n - C$ will tend to the integral over $-C$ and so the LHS tends to

$$\frac{\zeta(s)}{\Gamma(1-s)}.$$

Under the same condition on σ the series

$$\sum m^{s-1}$$

converges to $\zeta(1-s)$. Thus the RHS is a multiple of $\zeta(1-s)$.

Taking the limit gives the desired equation. A priori this is only valid for $\sigma < 0$ but if two meromorphic functions are equal on an open set they are equal everywhere. □

One can rewrite the functional equation. For example, replacing s by $1-s$ we have

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s).$$

One can also derive this using the functional equation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

We also have

Corollary 9.4. *The function*

$$\xi(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

is entire and satisfies $\xi(s) = \xi(1-s)$.

Proof. $\xi(s)$ is a meromorphic function on $\mathbb{C}$. The pole of $\zeta(s)$ at $s = 1$ cancels with $1-s$ and the poles of $\Gamma(s/2)$ cancel with the trivial zeroes of $\zeta(s)$. Thus $\xi(s)$ is an entire function.

Note that from the functional equation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

we get

$$\Gamma((1-s)/2)\Gamma((1+s)/2) = \frac{\pi}{\cos \pi s/2}.$$

Recall also Legendre's duplication formula

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2).$$

$$
\begin{aligned}
\xi(1-s) &= \frac{1}{2}(1-s)s\pi^{(s-1)/2}\Gamma((1-s)/2)\zeta(1-s) \\
&= \frac{1}{2}s(1-s)\pi^{(s-1)/2}\Gamma((1-s)/2)2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s) \\
&= \frac{1}{2}s(1-s)\pi^{-s/2}\zeta(s)2^{1-s}\pi^{-1/2}\Gamma(s)\Gamma((1-s)/2)\cos\frac{\pi s}{2} \\
&= \frac{1}{2}s(1-s)\pi^{-s/2}\zeta(s)\pi^{1/2}\Gamma(s)2^{1-s}\Gamma((1+s)/2)^{-1} \\
&= \frac{1}{2}s(1-s)\zeta(s)\pi^{-s/2}\Gamma(s/2) \\
&= \xi(s). \qquad \square
\end{aligned}
$$

We know from the series development of $\zeta(s)$ that there are no zeroes in the region $\sigma > 1$. Using the functional equation it follows that $\zeta(s)$ has no zeroes in the region $\sigma < 0$ apart from the trivial zeroes. So all of the zeroes belong to the strip $0 \leq \sigma \leq 1$. The **Riemann hypothesis** states that the only zeroes in the strip $0 \leq \sigma \leq 1$ belong to the line $\sigma = 1/2$.

It is known that there are no zeroes on the lines $\sigma = 0$ and $\sigma = 1$. It also known that at least one third of the zeroes lines on the line $\sigma = 1/2$.