8. Stirling Formula

Stirling's Formula is a classical formula to compute $n!$ accurately when n is large.

We will derive a version of Stirling's formula using complex analysis and residues. Recall the formula for the second logarithmic derivative of the gamma function:

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Let's start with the partial sum

$$\frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots + \frac{1}{(z+n)^2}.$$

We'd like to compute this as an integral. To this end, we'd like a function with residues

$$\frac{1}{(z+\nu)^2}$$

at the integral points ν. We pick

$$\Phi(w) = \frac{\pi \cot \pi w}{(z+w)^2}.$$

$w = u + iv$ is the variable of integration. For the time being we keep $z = x + iy$ fixed and we assume that $x > 0$.

We integrate around the rectangle whose vertical sides are $u = 0$ and $u = n + 1/2$ and whose horizontal sides are $v = \pm Y$. Call this contour K. Note that this contour contains a pole of $\Phi(w)$, at $w = 0$, so that we need to take the principal value:

$$\text{pr. v.}\ \frac{1}{2\pi i}\int_K \Phi(w)\, \mathrm{d}w = -\frac{1}{2z^2} + \sum_{m=0}^{n} \frac{1}{(z+m)^2}.$$

We will first let Y go to infinity and then n. Note that $\cot \pi w$ tends uniformly to $\pm i$ as v goes to ∞. Since $1/(z+w)^2$ goes to zero the integral over the horizontal sides goes to zero. Over the infinite vertical line $u = n + 1/2$, $\cot \pi w$ is bounded, and by periodicity this bound is independent of n. The integral over the line $u = n + 1/2$ is therefore at most a constant multiple of

$$\int_{u=n+1/2} \frac{\mathrm{d}w}{|w+z|^2}.$$

Note that on the line $u = n + 1/2$ we have

$$\bar{w} = 2n + 1 - w$$

and so

$$\frac{1}{i}\int_{u=n+1/2}\frac{\mathrm{d}w}{|w+z|^2}=\frac{1}{i}\int_{u=n+1/2}\frac{\mathrm{d}w}{(w+z)(2n+1-w+\bar{z})}.$$

We can calculate the last integral using residues. The poles are at $w=-z$ and $w=2n+1+\bar{z}$. Only the first $z=-w$ is to the left of the line $u=n+1/2$. The residue at $w=-z$ is

$$\frac{1}{2n+1+2x}.$$

The integral is therefore

$$\frac{2\pi}{2n+1+2x}.$$

This goes to zero as n goes to infinity.

It remains to deal with the principal value of the integral over the imaginary axis.

$$\frac{1}{2}\int_0^\infty \cot\pi iv\left[\frac{1}{(iv+z)^2}-\frac{1}{(iv-z)^2}\right]\mathrm{d}v=-\int_0^\infty \coth\pi v\frac{2vz}{(v^2+z^2)^2}\,\mathrm{d}v.$$

Putting all of this together we get

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)=\frac{1}{2z^2}+\int_0^\infty \coth\pi v\frac{2vz}{(v^2+z^2)^2}\,\mathrm{d}v.$$

We use the expression

$$\coth\pi v=1+\frac{2}{e^{2\pi v}-1}.$$

Now the integral from the first term is

$$\int_0^\infty\frac{2vz}{(v^2+z^2)^2}\,\mathrm{d}v=\left[-\frac{z}{(v^2+z^2)}\right]_0^\infty=\frac{1}{z}.$$

Thus

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)=\frac{1}{z}+\frac{1}{2z^2}+\int_0^\infty\frac{4vz}{(v^2+z^2)^2}\,\frac{\mathrm{d}v}{e^{2\pi v}-1},$$

where the integral is strongly very convergent.

If we restrict z to the right half plane we can integrate this formula.

$$\frac{\Gamma'(z)}{\Gamma(z)}=C+\log z-\frac{1}{2z}+\int_0^\infty\frac{2v}{(v^2+z^2)}\,\frac{\mathrm{d}v}{e^{2\pi v}-1},$$

where $\log z$ is the principal branch and C is a constant.

Note that if we restrict z to a compact subset of the right half plane then the integral converges uniformly and so we are allowed to differentiate under the integral sign.

We want to integrate once more. If we did this directly we would get $\tan(z/v)$ which is multi-valued. Instead we integrate by parts first,

$$\int_0^\infty \frac{2v}{(v^2+z^2)}\,\frac{\mathrm{d}v}{e^{2\pi v}-1} = \frac{1}{\pi}\int_0^\infty \frac{z^2-v^2}{(v^2+z^2)^2}\log(1-e^{-2\pi v})\,\mathrm{d}v.$$

Now we integrate with respect to z again to get

$$\log\Gamma(z) = C' + Cz + (z-\frac{1}{2})\log z + \frac{1}{\pi}\int_0^\infty \frac{z}{(v^2+z^2)^2}\log\frac{1}{1-e^{-2\pi v}}\,\mathrm{d}v.$$

Here C' is a new constant of integration and $C-1$ has been replaced by C. This formula means that $\log\Gamma(z)$ is single-valued on the right half plane and given by the expression on the right. We choose C' so that the LHS is real on the real axis.

To determine the constants C and C' we need to consider the integral

$$\frac{1}{\pi}\int_0^\infty \frac{z}{(v^2+z^2)^2}\log\frac{1}{1-e^{-2\pi v}}\,\mathrm{d}v.$$

We first check that $J(z)$ tends to zero as z goes to infinity but stays away from the imaginary axis. Suppose that we restrict z to the half plane $x \geq c > 0$. We break the integral into two parts

$$J(z) = \int_0^{\frac{|z|}{2}} + \int_{\frac{|z|}{2}}^\infty = J_1 + J_2.$$

In the first integral

$$|v^2+z^2| \geq |z|^2 - |z/2|^2 = \frac{3|z|^2}{4}$$

and so

$$|J_1| \leq \frac{4}{3\pi|z|}\int_0^\infty \log\frac{1}{1-e^{-2\pi v}}\,\mathrm{d}v.$$

In the second integral

$$|v^2+z^2| = |z-iv||z+iv| > c|z|$$

and so

$$|J_2| < \frac{1}{\pi c}\int_{|z|/2}^\infty \log\frac{1}{1-e^{-2\pi v}}\,\mathrm{d}v.$$

Since the integral of $\log\frac{1}{1-e^{-2\pi v}}$ is convergent J_1 and J_2 tend to zero as z goes to infinity.

To determine C we use the functional equation $\Gamma(z+1) = z\Gamma(z)$ or $\log\Gamma(z+1) = \log z + \log\Gamma(z)$, which is valid provided we stay in the right half plane. Hence

$$C'+Cz+C+(z+\frac{1}{2})\log(z+1)+J(z+1) = C'+Cz+(z+\frac{1}{2})\log z+J(z)$$

and so

$$C = \left(z + \frac{1}{2}\right) \log\left(1 + \frac{1}{z}\right) + J(z) - J(z+1).$$

Letting z go to infinity we get

$$C = -1.$$

Using the other functional equation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ we can deduce (with some work!) that $C' = \frac{1}{2}\log 2\pi$. Thus

$$\log \Gamma(z) = \frac{1}{2}\log 2\pi - z + (z - 1/2)\log z + J(z).$$

Equivalently

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} e^{J(z)}.$$

This is **Stirling's formula**. As $J(z)$ approaches 0 as z approaches infinity we can use Stirling's formula to estimate $n!$ for large values of n.

Theorem 8.1.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}t$$

for any $x > 0$.

Proof. Let

$$F(z) = \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}t.$$

Note that, integrating by parts,

$$F(z+1) = \int_0^\infty e^{-t} t^z \, \mathrm{d}t = z \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}t = zF(z).$$

It follows that $F(z)$ is in fact an entire function and that

$$\frac{F(z+1)}{\Gamma(z+1)} = \frac{F(z)}{\Gamma(z)},$$

so that the ratio is periodic with period one.

The goal is to apply Liouville's theorem to the ratio. The key point is to bound the absolute value of the ratio $|F/\Gamma|$ on some strip, say on $1 \leq x \leq 2$. For a start

$$|F(z)| \leq \int_0^\infty e^{-t} t^{x-1} \, \mathrm{d}t = F(x),$$

so that $F(z)$ is bounded in the strip. Now we apply Stirling's formula to find a lower bound for $|\Gamma(z)|$:

$$\log |\Gamma(z)| = \frac{1}{2}\log 2\pi - x + (x - 1/2)\log |z| - y \arg z + \operatorname{Re} J(z).$$

The only term that bothers us is $-y \arg z$ which can go to negative infinity, comparable to $-\pi|y|/2$. Thus $|F/\Gamma|$ grows no faster than $e^{\pi|y|/2}$.

Note that F/Γ is a function of the variable $q = e^{2\pi i z}$. It has singularities at $q = 0$ and $q = \infty$ but $|F/\Gamma|$ grows at most like $|q|^{-1/2}$ as q approaches zero and $|q|^{1/2}$ as q approaches infinity. Thus the singularities there are removable and F/Γ is constant by Liouville.

Since $F(1) = \Gamma(1) = 1$ it follows that $F(z) = \Gamma(z)$. $\square$