6. The Riemann Zeta function

Definition-Lemma 6.1. *The function*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (s = \sigma + it)$$

is called the ***Riemann zeta function****.*

$\zeta(s)$ *is a holomorphic function for* $\operatorname{Re} s > 1$.

Proof. Compare the sum

$$\sum_{n=1}^{\infty} n^{-s}$$

with the sum

$$\sum_{n=1}^{\infty} n^{-\sigma}$$

which converges uniformly for all $\sigma \geq \sigma_0$, where $\sigma_0 > 1$ is fixed. □

Enumerate the prime numbers in increasing order:

$$p_1, p_2, \dots.$$

Theorem 6.2. *For* $\sigma = \operatorname{Re} s > 1$

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty}(1 - p_n^{-s}).$$

Proof. First we check absolute convergence of the product. We have to consider convergence of the sum

$$\sum_{n=1}^{\infty} |p_n^{-s}| = \sum_{n=1}^{\infty} p_n^{-\sigma}.$$

If we compare this with

$$\sum_{n=1}^{\infty} n^{-s}$$

which converges uniformly for all $\sigma \geq \sigma_0$, where $\sigma_0 > 1$ is fixed we see that the product converges uniformly in the same range.

Thus for $\sigma > 1$ we have

$$\zeta(s)(1 - 2^{-s}) = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} (2n)^{-s} = \sum m^{-s}$$

where m runs over the odd integers.

Similarly, by inclusion-exclusion,

$$\zeta(s)(1 - 2^{-s})(1 - 3^{-s}) = \sum m^{-s}$$

where now m runs over the integers which are not divisible by 2 or by 3.

More generally, again by inclusion-exclusion

$$\zeta(s)(1-2^{-s})(1-3^{-s})\cdots(1-p_N^{-s}) = \sum m^{-s}$$

where now m runs over the integers which are not divisible by any of the primes up to p_N. The first term in the sum is 1 and the next one is p_{N+1}^{-s}. Therefore, as N tends to the infinity, the sum on the right tends to 1.

It follows that

$$\lim_{N\to\infty} \zeta(s)\prod_{i=1}^{N}(1-p_i^{-s}) = 1. \qquad \square$$

Corollary 6.3 (Euclid)**.** *There are infinitely many primes.*

Proof. We have

$$\zeta(s)\prod_{p}(1-p^{-s}) = 1$$

where the product runs over all primes. As s tends to one the Riemann zeta function tends to

$$\sum_{n=1}^{\infty} n^{-1}$$

which diverges. Thus

$$\prod_{p}(1-p^{-s})$$

tends to zero. This is only possible if the product is an infinite product. $\square$