5. Jensen formula

Theorem 5.1 (Jensen's Formula). *Let $f(z)$ be a holomorphic function for $|z| \leq \rho$.*

Then

$$\log|c| + h\log\rho = -\sum_{i=1}^{n} \log\left(\frac{\rho}{|a_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\,\mathrm{d}\theta,$$

where $a_1, a_2, \dots, a_n$ are the non-zero zeroes, repeated according to multiplicity, of f in the open disc $|z| < \rho$ and

$$f(z) = cz^h + \dots$$

is the power series expansion for $f(z)$.

Proof. We first prove this result under the hypotheses that $f(z)$ is nowhere zero in the closed disc $|z| \leq \rho$. Under these assumptions $\log|f(z)|$ is a harmonic function and the LHS is just $\log|f(0)|$.

Thus

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\,\mathrm{d}\theta.$$

Now suppose that $f(z)$ has zeroes on the circle $|z| = \rho$. We check that the same formula holds. If we replace $f(z)$ by

$$g(z) = \frac{f(z)}{z - \rho e^{i\theta_0}}$$

then $g(z)$ is a holomorphic function with one fewer zero than $f(z)$. By induction on the number of zeroes on the circle

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(\rho e^{i\theta})|\,\mathrm{d}\theta.$$

Now

$$\log|f(0)| = \log|g(0)| - \log\rho,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(\rho e^{i\theta})|\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\,\mathrm{d}\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|\rho e^{i\theta} - \rho e^{i\theta_0}|\,\mathrm{d}\theta.$$

It suffices then to show that

$$\log\rho = \frac{1}{2\pi}\int_0^{2\pi} \log|\rho e^{i\theta} - \rho e^{i\theta_0}|\,\mathrm{d}\theta.$$

Equivalently we want

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - e^{i\theta_0}|\,\mathrm{d}\theta = 0.$$

By symmetry this integral does not depend on θ_0. So we just have to show that

$$\int_0^{2\pi} \log|1 - e^{i\theta}| \, \mathrm{d}\theta = 0.$$

But we actually showed this when we computed the value of

$$\int_0^{\pi} \log \sin x \, \mathrm{d}x$$

using contour integration.

Now suppose that $f(z)$ has zeroes $a_1, a_2, \dots, a_n$ (repeated according to multiplicity) but $f(z)$ is non-zero at the origin. Let

$$F(z) = f(z) \prod_{i=1}^{n} \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)}.$$

Then $F(z)$ doesn't vanish anywhere in the disc $|z| < \rho$ and $|F(z)| = |f(z)|$. Thus

$$\log|F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(\rho e^{i\theta})| \, \mathrm{d}\theta.$$

It follows that

$$\log|f(0)| = -\sum_{i=1}^{n} \log\left(\frac{\rho}{|a_i|}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log|f(\rho e^{i\theta})| \, \mathrm{d}\theta.$$

Finally we need to consider the general case when $f(z)$ is possibly zero at the origin. Let

$$g(z) = f(z) \left(\frac{\rho}{z}\right)^h = c\rho^h + \dots.$$

Then $g(z)$ doesn't vanish at the origin $|g(z)| = |f(z)|$ on the circle $|z| = \rho$ and

$$\log|g(0)| = \log|c| + h \log \rho.$$ □

Corollary 5.2 (Poisson-Jensen formula)**.** *Let $f(z)$ be a holomorphic function for $|z| \leq \rho$ such that $f(z) \neq 0$.*

Then

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)}\right| + \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \log|f(\rho e^{i\theta})| \, \mathrm{d}\theta,$$

where $a_1, a_2, \dots, a_n$ are the non-zero zeroes, repeated according to multiplicity, of f in the open disc $|z| < \rho$.

Proof. Apply Jensen's formula to the function $F(z)$ appearing in the proof of (5.1). □