3. Harmonic functions

Recall:

Definition 3.1. *Let $U \subset \mathbb{C}$ be a region.*
*We say that $u\colon U \longrightarrow \mathbb{R}$ is **harmonic**, if it is $\mathcal{C}^2$ and*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The set of Harmonic functions is a vector space. The simplest harmonic functions are linear functions $ax + by + c$, where a, b and c are real.

Suppose that we introduce polar coordinates (r, θ). Then we get the equation

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

It follows that $\log r$ is a harmonic function and any harmonic function which only depends on r must be of the form $a \log r + b$.

Recall that if u is harmonic then u is locally the real part of a holomorphic function. The imaginary part v is called the harmonic conjugate. Unfortunately the harmonic conjugate is not unique, nor is it necessarily globally defined. Consider for example $U = \mathbb{C}^*$ and $u = \log r$.

If u is harmonic, then

$$f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y},$$

is holomorphic. If we put

$$U = \frac{\partial u}{\partial x} \qquad \text{and} \qquad V = -\frac{\partial u}{\partial y},$$

then

$$\begin{aligned} \frac{\partial U}{\partial x} &= \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial V}{\partial y} \\ \frac{\partial U}{\partial y} &= \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial V}{\partial x}. \end{aligned}$$

In fact we can put this in the form of differentials

$$f\,\mathrm{d}z = \left(\frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y\right) + i\left(-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y\right).$$

Note that in this expression the real part is the differential of u,

$$\mathrm{d}u = \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y.$$

Suppose that v is the harmonic conjugate of u. Then the imaginary part can be written as

$$\mathrm{d}v = \frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y = -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y.$$

However the harmonic conjugate need not exist and even if it does it is not unique. For this reason we write

$$*\mathrm{d}u = -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y,$$

which we call the **conjugate differential** of u. Thus

$$f\,\mathrm{d}z = \mathrm{d}u + i\,*\mathrm{d}u.$$

Now the integral of $f\,\mathrm{d}z$ around any cycle homologous to zero, vanishes, by the general form of Cauchy's Theorem. The integral of $\mathrm{d}u$ around any cycle is zero, as $\mathrm{d}u$ is exact. Thus

$$\int_\gamma *\mathrm{d}u = \int_\gamma -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y = 0,$$

for any cycle homologous to zero. There is an interesting generalisation of this result to pairs of harmonic functions u_1 and u_2:

Theorem 3.2. *If u_1 and u_2 are harmonic in a region U then*

$$\int_\gamma u_1 *\mathrm{d}u_2 - u_2 *\mathrm{d}u_1 = 0,$$

for every cycle homologous to zero.

Proof. It suffices to prove this is in the very special case when $\gamma = \partial R$ is the boundary of a rectangle. In R we may find conjugate harmonic functions v_1 and v_2. In this case

$$u_1 *\mathrm{d}u_2 - u_2 *\mathrm{d}u_1 = u_1\,\mathrm{d}v_2 - u_2\,\mathrm{d}v_1 = u_1\,\mathrm{d}v_2 + v_1\,\mathrm{d}u_2 - \mathrm{d}(u_2 v_1).$$

Now $\mathrm{d}(u_2v_1)$ is an exact differential and $u_1\,\mathrm{d}v_2 + v_1\,\mathrm{d}u_2$ is the imaginary part of

$$(u_1 + iv_1)\,\mathrm{d}(u_2 + iv_2).$$

Integrating an exact differential over γ is zero. On the other hand

$$(u_1 + iv_1)\,\mathrm{d}(u_2 + iv_2) = F(z)f(z)\,\mathrm{d}z$$

for appropriate holomorphic functions $F(z)$ and $f(z)$, so that integrating the product above is zero, since the product of two holomorphic functions is holomorphic. □

Theorem 3.3. *If u is harmonic in the annulus $\rho_1 < |z| < \rho_2$ then there are constants α and β such that*

$$\frac{1}{2\pi}\int_{|z|=r} u\,\mathrm{d}\theta = \alpha\log r + \beta.$$

If further u is harmonic in the whole disc, then $\alpha = 0$, so that the integral is constant.

Proof. We apply (3.2) with $u_1 = \log r$ and $u_2 = u$. Let γ be the cycle obtained by describing the two circles $|z| = r_i$ in the opposite orientation, where $\rho_1 < r_1 < r_2 < \rho_2$. Then γ is homologous to zero, so that

$$\int_\gamma u_1 *\mathrm{d}u_2 - u_2 *\mathrm{d}u_1 = 0.$$

Now

$$*\mathrm{d}u = r\frac{\partial u}{\partial r}\,\mathrm{d}\theta,$$

on the circle $|z| = r$, so that we have

$$\log r_1 \int_{|z|=r_1} r_1\frac{\partial u}{\partial r}\,\mathrm{d}\theta - \int_{|z|=r_1} u\,\mathrm{d}\theta = \log r_2 \int_{|z|=r_2} r_2\frac{\partial u}{\partial r}\,\mathrm{d}\theta - \int_{|z|=r_2} u\,\mathrm{d}\theta.$$

It follows that the expression

$$\int_{|z|=r} u\,\mathrm{d}\theta - \log r\int_{|z|=r} r\frac{\partial u}{\partial r}\,\mathrm{d}\theta,$$

is independent of r, in the annulus. On the other hand, since

$$\int_\gamma *\mathrm{d}u = 0$$

if we run the same argument then we see that

$$\int_{|z|=r} r\frac{\partial u}{\partial r}\,\mathrm{d}\theta,$$

is also constant in the annulus. □

Corollary 3.4. *Let u be a harmonic function on U.*
Then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta,$$

for any $z_0 \in U$, where r is sufficiently small.

Proof. Apply (3.2) to a circle centred at z_0, sufficiently small so that it is contained in U. In this case $\alpha = 0$ and β is the value of u at z_0. □

Corollary 3.5 (Maximum principle)**.** *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition. Thus the maximum and the minimum of a harmonic function on a compact set E are achieved on the boundary.*

Note that the maximum principle has an interesting consequence. A continuous function u on a closed set, which is harmonic on the interior, is determined by its values on the boundary.

Indeed suppose that u_1 and u_2 are two continuous functions on E, which are harmonic on the interior and which agree on the boundary. Then $u_1 - u_2$ is a harmonic function which is zero on the boundary. On the other hand, the maximum and minimum value is taken on the boundary, so that the maximum and minimum of $u_1 - u_2$ is zero. But then $u_1 = u_2$.