2. The Gamma function

The zeroes of the function $\sin \pi z$ are the integers and it is the simplest function with this property. How about holomorphic functions whose zeroes are the positive (or negative) integers? The simplest choice of such a function is given by the canonical product:

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Obviously $G(z)$ is zero at all of the negative integers. As usual we throw in the exponential term to induce convergence.

On the other hand $G(-z)$ has zeroes at all of the positive integers. It follows that the ratio between the product $zG(z)G(-z)$ and $\sin \pi z$ is a function with no zeroes nor poles, so that it is the exponential of a function. In fact we showed in 220A, Lecture 24 that

$$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Thus

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi}.$$

Since the construction of $G(z)$ is so simple, we expect it to have some interesting properties. Note that $G(z-1)$ has the same zeroes as $G(z)$ as well as a zero at 0. So we can write

$$G(z-1) = ze^{\gamma(z)}G(z),$$

for some entire function $\gamma(z)$. To determine $\gamma(z)$ take the logarithmic derivative of both sides:

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right).$$

Let's take the LHS and replace n by $n+1$:

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n}\right) &= \frac{1}{z} - 1 + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n+1}\right) \\ &= \frac{1}{z} - 1 + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right).\end{aligned}$$

The last series sums to 1 and so $\gamma'(z) = 0$. It follows that $\gamma(z)$ is a constant. Let's denote this constant by γ, so that

$$G(z-1) = ze^{\gamma}G(z).$$

To determine the constant γ, plug in $z = 1$:

$$1 = G(0) = e^{\gamma}G(1).$$

Therefore

$$e^{-\gamma} = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)e^{-1/n}.$$

Now the nth partial product is

$$(n+1)e^{-(1+1/2+1/3+\cdots+1/n)},$$

and so

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right).$$

The constant γ is called **Euler's constant**.

$$\gamma \approx .57722.$$

If we set $H(z) = G(z)e^{\gamma z}$ then

$$H(z-1) = zH(z).$$

Thus

$$\Gamma(z) = \frac{1}{zH(z)}$$

satisfies

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1},$$

or better

$$\Gamma(z+1) = z\Gamma(z).$$

Γ is called **Euler's gamma function**.

We have

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)^{-1}e^{z/n}.$$

Note that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

$\Gamma(z)$ is a meromorphic function with poles at $z = 0, -1, -2, \ldots,$ and no zeroes.

Note that $\Gamma(1) = 1$, $\Gamma(2) = 1\Gamma(1) = 1$, $\Gamma(3) = 2\Gamma(2) = 2$, $\Gamma(4) = 3\Gamma(2) = 6$ and in general $\Gamma(n) = (n-1)!$. We can also see that

$$\Gamma(1/2) = \sqrt{\pi}.$$

To go further, it is useful to write down the second logarithmic derivative:

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}.$$

For example

$$\Gamma(z)\Gamma(z+1/2) \qquad \text{and} \qquad \Gamma(2z)$$

have the same poles. We have

$$\begin{aligned}
\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right) &= \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} + \sum_{n=0}^{\infty} \frac{1}{(z+n+1/2)^2} \\
&= 4\left[\sum_{n=0}^{\infty} \frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2z+2n+1)^2}\right] \\
&= 4\sum_{m=0}^{\infty} \frac{1}{(2z+m)^2} \\
&= 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right).
\end{aligned}$$

If we integrate then we get

$$\Gamma(z)\Gamma(z+1/2) = e^{az+b}\Gamma(2z),$$

where a and b are constants to be determined. Substituting $z = 1/2$ and $z = 1$ we make use of the known values

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(1) = 1, \qquad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}, \qquad \text{and} \qquad \Gamma(2) = 1.$$

This gives

$$\begin{aligned}
a/2 + b &= 1/2 \log \pi \\
a + b &= 1/2 \log \pi - \log 2.
\end{aligned}$$

It follows that

$$a = -2\log 2 \qquad \text{and} \qquad b = 1/2\log\pi + \log 2.$$

Putting all of this together we get

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2).$$

This is known as Legendre's (duplication) formula.