## 16. Maps between manifolds

**Definition 16.1.** *Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces. We say that $f$ is a* ***local homeomorphism*** *if for every point $x \in X$ there is an open neighbourhood $U$ of $x$ such that $V = f(U)$ is open and $f|_U\colon U \longrightarrow V$ is a homeomorphism.*

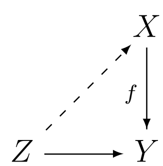
*We say that $f$ is an* ***unramified cover*** *if for every point $y \in Y$ we may find an open neighbourhood $V$ of $y$, such that the inverse image $f^{-1}(V)$ is a disjoint union of open subsets $U$ such that $f|_U\colon U \longrightarrow V$ is a homeomorphism.*

Every unramified cover is a local homeomorphism. However:

**Example 16.2.** *let $Y = \mathbb{C}$ and $X = \mathbb{C}^* = \mathbb{C} - \{0\}$. Then the inclusion of $X$ into $Y$ is a local homeomorphism but it is not an unramified cover.*

We will need some basic theory of covering spaces.

**Definition 16.3.** *Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces. Let $Z$ be any topological space and let $g\colon Z \longrightarrow Y$ be any continuous map. A* ***lift*** *of $g$ to $X$ is a continuous map $h\colon Z \longrightarrow X$ such that $f \circ h = g$.*




In general lifts need neither exist nor are they even unique:

**Example 16.4.** *Let $f\colon X = \mathbb{C} - \{0\} \longrightarrow \mathbb{C} = Y$ be the natural inclusion. Then we cannot lift the identity map $g\colon Y \longrightarrow Y$ to $X$.*

**Example 16.5.** *Let $X$ and $Y$ be the spaces constructed in (13.2.2). Then there is a natural continuous map $f\colon Y \longrightarrow X$ (note that we have switched $X$ and $Y$, both in terms of (13.2.2) and* (16.1)*). However the identity map $g\colon X \longrightarrow X$ is a continuous map which can be lifted in two different ways.*

However non-uniqueness is quite pathological:

**Lemma 16.6.** *Let $f\colon X \longrightarrow Y$ be a local homeomorphism of topological spaces. Let $Z$ be a connected topological space and let $g\colon Z \longrightarrow Y$ be a continuous map. Let $h_i\colon Z \longrightarrow X$, $i = 1, 2$ be any two maps which lift $g$ to $X$.*

*If $X$ is Hausdorff and there is a point $z_0 \in Z$ such that $h_1(z_0) = h_2(z_0)$ then $h_1 = h_2$.*

*Proof.* Let

$$E = \{\, z \in Z \mid h_1(z) = h_2(z) \,\},$$

be the set of points where $h_1$ and $h_2$ are equal. Then $z_0 \in E$ so that $E$ is non-empty by assumption. As $X$ is Hausdorff, the diagonal $\Delta \subset X \times X$ is closed. Now $E$ is the inverse image of $\Delta$ by the continuous map $h_1 \times h_2 \colon Z \longrightarrow X \times X$, so that $E$ is closed as $X$ is Hausdorff.

Suppose that $z \in Z$. Let $x = h_i(z)$. Then we may find an open neighbourhood $U$ of $x$ and $V = f(U)$ of $y = f(x) = g(z)$, such that $f|_U \colon U \longrightarrow V$ is a homeomorphism. Since $h_i$ are continuous, $W_i = h_i^{-1}(U)$ is open. Let $W = W_1 \cap W_2$. As

$$f \circ h_1 = g = f \circ h_2,$$

and $f|_U$ is injective, $W \subset E$. But then $E$ is open and closed, so that $E = Z$ as $Z$ is connected. □

**Lemma 16.7.** *Let $f \colon X \longrightarrow Y$ be an unramified cover of manifolds. Let $\gamma \colon [0,1] \longrightarrow Y$ be a path and suppose $x \in X$ is point of $X$ such that $f(x) = y = \gamma(0)$.*

*Then $\gamma$ has a unique lift to a map*

$$\psi \colon [0,1] \longrightarrow X,$$

*such that $\psi(0) = x$.*

*Proof.* For every point $y$ of the image of $\gamma$, we may find a connected open neighbourhood $V = V_y \subset Y$ of $y$ such that $f^{-1}(V)$ is a disjoint union of open subsets all of which are homeomorphic to $V$ via $f$. Since the image of $\gamma$ is compact, we may cover the image by finitely many of these open subsets $V_1, V_2, \dots, V_k$. Let $W_j \subset [0,1]$ be the inverse image of $V_j$, and let

$$I_j = \bigcup_{i \leq j} W_i.$$

Then $I_j$ is an open subset of $[0,1]$ which is an interval containing 0. We define $\psi_j \colon I_j \longrightarrow X$ by induction on $j$. Suppose that we have defined $\psi_k$, $k \leq j$. Pick $t \in W_j \cap W_j \subset I_i \cap W_{j+1}$. Let $x' = \psi_j(t) \in X$ (if $j = 0$ then we take $x' = x$). Pick $U \subset X$ containing $x'$ such that $U$ is a connected component of $f^{-1}(V_{j+1})$ which maps homeomorphically down to $V_{j+1}$. Then we may clearly extend $\psi_j$ to $\psi_{j+1}$. Uniqueness follows from (16.6). □

**Definition 16.8.** *Let $f \colon X \longrightarrow Y$ be a local homeomorphism. We say that $f$ has the* ***lifting property*** *if given any continuous map $\gamma \colon [0,1] \longrightarrow Y$ and a point $x \in X$ such that $f(x) = y = \gamma(0)$, then we may lift $\gamma$ to $\psi \colon [0,1] \longrightarrow X$ such that $\psi(0) = x$.*

**Theorem 16.9.** *Let $f\colon X \longrightarrow Y$ be a local homeomorphism of manifolds.*

*Then $f$ has the lifting property if and only if $f$ is an unramified cover.*

We have already shown one direction of (16.9). To prove the other direction we need the following basic result:

**Theorem 16.10.** *Let $f\colon X \longrightarrow Y$ be a local homeomorphism of manifolds. Suppose that we have a homotopy $G\colon [0,1]\times[0,1] \longrightarrow Y$ between $\gamma_0$ and $\gamma_1$. Suppose also that we can lift $\gamma_s\colon [0,1] \longrightarrow Y$, defined by $\gamma_s(t) = \gamma(s,t)$ to $\psi_s\colon [0,1] \longrightarrow X$.*

*If the function $\sigma\colon [0,1] \longrightarrow X$ given by $\sigma(s) = \psi_s(0)$ is continuous then we can lift $G$ to a continuous map $H\colon [0,1]\times[0,1] \longrightarrow X$.*

*Proof.* The definition of

$$H\colon [0,1]\times[0,1] \longrightarrow X,$$

is clear; given $(s,t) \in [0,1]^2$, $H(s,t) = \psi_s(t)$. The only thing we need to check is that $H$ is continuous. Note that $\sigma(s) = H(s,0)$.

Given $(s,t) \in [0,1]^2$, let $x = H(s,t)$. Since $f$ is a local homeomorphism, we may find an open subset $U = U_{s,t}$ such that $V = f(U)$ is open and $f|_U\colon U \longrightarrow V$ is a homeomorphism. Let $W = G^{-1}(V)$. Then $W = W_{s,t}$ is an open neighbourhood of $(s,t)$, such that $H(s,t) \in U$ and $G(W) = V$. By compactness of $[0,1]^2$, we may find a finite subcover.

It follows that we may subdivide $[0,1]^2$ into finitely many squares in such a way that each square is mapped by $G$ into an open subset $V$ of $Y$ such that there is an open subset $U$ of $X$ with the property that $f|_U\colon U \longrightarrow V$ is a homeomorphism and there is a point $(s,t)$ belonging to the square such that $H(s,t) \in U$.

Suppose that the subdivision is given by $k$, so that we subdivide each $[0,1]$ into the intervals $[i/k,(i+1)/k]$. By an obvious induction, it suffices to prove that

$$H|_{[i/k,(i+1)/k]\times[j/k,(j+1)/k]},$$

is continuous, given that $\sigma(t) = H(i/k,t)$ is continuous. Replacing the square

$$[i/k,(i+1)/k]\times[j/k,(j+1)/k],$$

by $[0,1]^2$ and $Y$ by $V$, we are reduced to proving that $H$ is continuous, given that there is an open subset $U$ of $X$ which maps homeomorphically down to $Y$ and such that there is a single point $(s_0,t_0) \in [0,1]^2$ such that $H(s_0,t_0) \in U$.

Since $f|_U\colon U \longrightarrow Y$ is a homeomorphism, we may lift $\gamma_{s_0}$ to a map $\psi_0\colon [0,1] \longrightarrow U$. By uniqueness it follows that $H(s_0,t) \in U$ for all $t \in [0,1]$. By uniqueness of the lift of $\gamma$, where $\gamma(t) = G(0,t)$, $H(0,t) \in U$

for all $t \in [0,1]$. Finally by uniqueness of the lift of $\gamma_s(t)$, it follows that $H(s,t) \in U$ for all $(s,t) \in [0,1]^2$. But then $G = (f|_U)^{-1} \circ H$ is certainly continuous. □

To prove (16.9), we start with a seemingly special case:

**Proposition 16.11.** *Let $f\colon X \longrightarrow Y$ be a local homeomorphism of connected manifolds.*

*If $f$ has the lifting property and $Y$ is simply connected then $f$ is an isomorphism.*

*Proof.* Pick $a \in X$ and let $b = f(a) \in Y$.

Pick $y \in Y$. Since $Y$ is a connected manifold it is path connected. Pick a path $\gamma\colon [0,1] \longrightarrow Y$ such that $b = \gamma(0)$ and $y = \gamma(1)$. Let $\psi\colon [0,1] \longrightarrow X$ be a lifting of $\gamma$ and let $x = \psi(1)$. Then $f(x) = y$. Thus $f$ is surjective.

Suppose that $x_0$, $x_1 \in X$, with $y = f(x_0) = f(x_1)$. Pick paths $\psi_i\colon [0,1] \longrightarrow X$ such that $a = \psi_i(0)$ and $x_i = \psi_i(1)$. Let $\gamma_i = f \circ \psi_i$. As $Y$ is simply connected, we may pick a homotopy $G$ between $\psi_0$ and $\psi_1$ fixing $y$. By (16.10) we may lift $G$ to a continuous function $H$, such that $H(0,0) = x$. Since the function $\psi(t) = G(0,t)$ is continuous, the composition $f \circ \psi$ is constant (equal to $y$) and the fibre $f^{-1}(y)$ is discrete, it follows that $G(0,t) = x$ is constant. But then $x_1 = G(0,1) = G(0,0) = x_0$. Thus $f$ is injective.

As $f$ is a local homeomorphism, it follows that $f$ is a homeomorphism. □

*Proof of* (16.9). Suppose that $f$ has the lifting property.

Pick $y \in Y$ and let $V$ be a simply connected neighbourhood of $y$. Let $U$ be a connected component of the inverse image. Then $f|_U\colon U \longrightarrow V$ has the lifting property and by (16.11) it is a homeomorphism. □