14. Subharmonic functions

Definition 14.1. *Let $u\colon U \longrightarrow \mathbb{R}$ be a continuous function. We say that u satisfies the* ***mean-value property*** *if*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,\mathrm{d}\theta,$$

whenever the disk $|z - z_0| \leq r$ is contained in U

Note that the mean-value property implies the maximum principle. In fact it suffices that the mean-value property holds in a small ball about every point of $z_0 \in U$, whose radius depends on z_0.

Theorem 14.2. *A continuous function $u(z)$ on a domain U satisfies the mean-value property if and only if it is harmonic.*

Proof. If u is harmonic we have already seen that it must satisfy the mean-value property.

Now suppose that u satisfies the mean-value property. Let v be any harmonic function. Then the difference $u - v$ also satisfies the mean-value property. In particular the difference satisfies the maximum principle.

Pick a ball contained in U for which u satisfies the mean-value property. Let U be the restriction of u to the boundary of this ball and let $v = P_U$ be the Poisson integral. Then v is a harmonic function which agrees with u on the boundary of the ball. Thus the difference $u - v$ satisfies the mean-value property and is zero on the boundary. By the maximum principle $u - v$ is zero so that $u = v$ is harmonic on the ball. Thus u is harmonic. $\square$

Using (14.2) it is clear that we could have defined a function to be harmonic if and only if it satisfies the mean-value property, so that we can give a definition of harmonic which makes no reference to the existence of partial derivatives.

Theorem 14.3 (Harnack's inequality)**.** *Let u be a harmonic function on a disc centred at the origin of radius ρ and let z be a point of distance r away from the origin.*

If $u(z) \geq 0$ then

$$\frac{\rho - r}{\rho + r}u(0) \leq u(z) \leq \frac{\rho + r}{\rho - r}u(0).$$

Proof. Note that

$$\frac{\rho - r}{\rho + r} \leq \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho + r}{\rho - r}.$$

Thus

$$|u(z)| \leq \frac{1}{2\pi}\frac{\rho+r}{\rho-r}\int_0^{2\pi} |u(\rho e^{i\theta})|\, d\theta.$$

If further $u(\rho e^{i\theta}) \geq 0$ then we can apply the first inequality as well to obtain

$$\frac{1}{2\pi}\frac{\rho-r}{\rho+r}\int_0^{2\pi} u\, d\theta \leq u(z) \leq \frac{1}{2\pi}\frac{\rho+r}{\rho-r}\int_0^{2\pi} u\, d\theta.$$

Now use the fact that the integral is equal to the value of u at the origin. □

Theorem 14.4 (Harnack's principle)**.** *Let $u_n(z)$ be a harmonic function on a region U_n. Suppose that U is a region such that every point of U has a neighbourhood which is contained in all but finitely many U_n and that $u_n(z) \leq u_{n+1}(z)$ for all but finitely many n on the same neighbourhood.*

Then either $u_n(z)$ tends uniformly to ∞ on compact subsets or it tends to a harmonic function $u(z)$, uniformly on compact subsets.

Proof. Suppose first that there is a $z_0 \in U$ such that $u_n(z_0)$ tends to ∞. Then we may find $r > 0$ and m such that the functions u_n, $n > m$ are non-decreasing for $|z - z_0| < r$. Applying the left side of Harnack's inequality to $u = u_n - u_m$, we see that u_n tends uniformly to ∞ in the disk $|z - z_0| \leq r/2$. Similarly if the limit at z_0 is finite, then the right side of Harnack's inequality implies that u_n converges uniformly to a finite number on the disk $|z - z_0| \leq r/2$.

Thus the set of points where u_n tends either to zero or to infinity are both open. As U is connected one such set must be empty. If the limit is infinite, uniformity follows by the usual compactness argument.

Now suppose that the limit is finite everywhere. Then

$$u_{n+p}(z) - u_n(z) \leq 3(u_{n+p}(z_0) - u_n(z_0)) \qquad \text{for} \qquad |z - z_0| \leq r/2,$$

and $n + p \geq n \geq m$. Thus convergence at z_0 implies uniform convergence in a neighbourhood of z_0. Thus we get that u_n tends to a limit u uniformly on compact subsets. But $u(z)$ must be harmonic by Poisson's formula. □

Note that Laplace's equation in one variable reduces to requiring that the second derivative is zero. In this case the solutions to Laplace's equation are the linear functions. A function on an interval, which is less than the linear function with the same values at the endpoints of the interval, is called convex. A subharmonic function is any function of two variables with the same property. Any function which is less than a harmonic function with the same boundary values is called subharmonic. Since this definition is a little unmanageable, we are lead to:

Definition 14.5. *A continuous function $v(z)$ on a region U is said to be **subharmonic** in U if for any harmonic function u defined on any open subset $U' \subset U$ the difference $v-u$ satisfies the maximum principle in U' (that is, if $v-u$ has a maximum then it is constant).*

Note that the condition that v is subharmonic is local in nature. In fact if we say that v is subharmonic at z_0 if it is subharmonic in a neighbourhood of z_0, then v is subharmonic if and only if it is subharmonic at each point of U.

Note that a harmonic function is subharmonic. Note that if v is $\mathcal{C}^2$ and z_0 is a maximum of $v-u$ then the partials of $v-u$ at z_0 vanish and the 2nd derivative is non-positive, so that $\Delta v = \Delta(v-u) \leq 0$. In fact this is enough to characterise subharmonic functions whose second partials exist:

Lemma 14.6. *Let v be a continuous function whose second partial derivatives exist on U.*

Then v is subharmonic if and only if $\Delta v \geq 0$ on U.

Proof. Exercise left to the reader. □

Note however that there are many subharmonic functions whose derivatives do not exist.

Theorem 14.7. *A continuous function $v(z)$ is subharmonic if and only if it satisfies the inequality*

$$v(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta,$$

for every disk $|z-z_0| \leq r$ contained in U.

Proof. Suppose that v satisfies the inequality. Let u be a harmonic function. Suppose that z_0 is a maximum of $v-u$. Then $v-u$ also satisfies the inequality. Replacing v by $v-u$, we may suppose that z_0 is a maximum of v. Suppose that the disk $|z-z_0| < r$ is contained in U. If $v(z) < v(z_0)$ then the integral on the right must be less than $v(z_0)$, since it is an average, a contradiction. Thus v is constant on the boundary of the disk, whence it is constant on the disc, by varying the radius r. But then v must be constant, so that v is subharmonic.

Now suppose that v is subharmonic. Let V be the restriction of v to the boundary of the disk $|z-z_0| < r$ and let P_V be the corresponding Poisson integral. Then P_V is harmonic so that $v - P_V$ satisfies the maximum principle on the disk. It is zero on the boundary of the disk, so that

$$v \leq P_V$$

in the interior. □

Here are some basic properties of subharmonic functions:

Lemma 14.8. *Suppose that* $v = v_1$ *and* v_2 *are subharmonic. Then so is*

(1) kv*, for any constant* $k \geq 0$*,*
(2) $v_1 + v_2$*,*
(3) $w = \max(v_2, v_2)$*, and*
(4) the function v' *which is equal to* v *outside a disc and* P_v *inside a disc.*

Proof. (1) and (2) follow from (14.7).

Let u be any harmonic function such that $w - u$ has a maximum at z_0. Replacing v_i by $v_i - u$ we may assume that $u = 0$. Suppose that $w(z_0) = v_1(z_0)$. Then

$$v_1(z) \leq w(z) \leq w(z_0) = v_1(z_0).$$

Thus $v_1(z)$ is constant. Now either $w(z) = v_1(z)$ in a neighbourhood of z_0 or $w(z_0) = v_2(z_0)$. But then $v_2(z)$ is also constant in a neighbourhood of z_0 and either way it follows that $w(z)$ is constant in a neighbourhood of z_0. But then $w(z)$ satisfies the maximum principle. Thus w is subharmonic. This is (3).

Note that v' is continuous since P_v is equal to v on the boundary of the disk. On the other hand as P_v is a harmonic function, it is certainly subharmonic. We can check that v' is subharmonic at every point z_0. Suppose that z_0 is a maximum of v'. If z_0 belongs to the interior of the disc we are done since P_v is subharmonic. If z_0 belongs to the boundary of the disc then z_0 is a local maximum of v. But then v is constant and so P_v is constant. Thus is (4). □