Definition 13.1. *Let X be a topological space.*

We say that X is a ***topological manifold****, if*

1. *X is Hausdorff,*
2. *X is 2nd countable (that is, there is a base for the topology which is countable),*
3. *for every point $x \in X$, there is an open neighbourhood $U \subset X$, an open subset $V \subset \mathbb{R}^n$ and a homeomorphism $f\colon U \longrightarrow V$.*

The ***dimension*** *of X at x is equal to n. The map f is called a* ***chart****.*

The dimension is locally constant, so that the dimension is constant on the connected components of X. The space of topological manifolds and continuous maps forms a full subcategory of the category of topological spaces and continuous maps.

Example 13.2.

1. *Any open subset of $\mathbb{R}^n$ is a topological manifold of dimension n. (A countable base of $\mathbb{R}^n$ is given by balls of rational radius with rational coordinates and any subset of a 2nd countable topological manifold is 2nd countable).*
2. *Let Y be the disjoint union of two copies of $(0, 1)$. Let $\sim$ be the equivalence relation where the two corresponding points of the interval are equivalent, except the point $1/2$. Let $X = Y/\sim$ be the quotient. Then X is 2nd countable, and it is locally homeomorphic to $\mathbb{R}$ but it is not Hausdorff.*
3. *Let $Y = \omega_1 \times [0, 1) - \{(0, 0)\}$. Then Y is a topological space with the order topology, called the very long ray. Let X be the space obtained by joining two copies of Y together, along the sets $(0, 1) \times \{0\}$, but using the map*
$$t \longrightarrow 1 - t.$$
Then X is called the very long line. X (and indeed Y) is Hausdorff and locally homeomorphic to $\mathbb{R}$, but it is not 2nd countable.

We are interested in putting extra structure on a topological manifold. To this end, we need the following:

Definition 13.3. *Let X be a topological manifold. An* ***atlas*** *is an open cover $\{U_\alpha\}$ by charts $h_\alpha\colon U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^n$. Given α and β, the transition functions $g_{\alpha\beta}\colon V_{\alpha\beta} \longrightarrow V_{\beta\alpha}$ are the composition of $h_\alpha^{-1}\colon V_{\alpha\beta} \longrightarrow U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $h_\beta\colon U_{\alpha\beta} \longrightarrow V_{\beta\alpha}$ where $V_{\alpha\beta} = h_\alpha(U_{\alpha\beta})$.*

Definition 13.4. *We say that a topological manifold X is $\mathcal{C}^p$ if there is an atlas such that the transition functions are $\mathcal{C}^p$.*

A particularly interesting example is when $p = \infty$, in which case we sometimes say that X is a smooth manifold. Suppose that we are given a continuous map $f\colon M \longrightarrow N$ of $\mathcal{C}^p$-manifolds. Using the charts we can say what it means for f to be $\mathcal{C}^p$. In particular we get a category of $\mathcal{C}^p$-manifolds.

Definition 13.5. *A* ***Riemann surface*** *X is a topological space, which has an atlas where $V_\alpha \subset \mathbb{C}$ and the transition functions are holomorphic.*

As above we get a category:

Definition 13.6. *Let $f\colon X \longrightarrow Y$ be a continuous map of Riemann surfaces. We say that f is* ***holomorphic*** *if there are atlases on X and Y such that the induced map between charts is holomorphic.*

Note also that any Riemann surface is naturally a $\mathcal{C}^\infty$-manifold of dimension two, that is, a Riemann surface is a smooth surface. Note also that a Riemann surface has a natural orientation. Indeed multiplication by i determines the difference between left and right. Thus a Riemann surface is an oriented surface. Since a map between open subsets of $\mathbb{C}$ is holomorphic if and only if it is conformal and preserves orientation, in fact a Riemann surface is the same as a conformal structure on an oriented surface (intuitively a conformal structure is nothing more than the specification of angles between germs of arcs).

Example 13.7. *Any open subset of $\mathbb{C}$ is a Riemann surface.*

Example 13.8. *The Riemann sphere is a Riemann surface. In this case the transition functions are $z \longrightarrow 1/z$. The Riemann sphere is naturally isomorphic to $\mathbb{P}^1$.*

To show that there are in fact many compact Riemann surfaces, let us see how to associate to a polyhedral surface $\mathcal{P}$ in $\mathbb{R}^3$ a Riemann surface X. By a polyhedral surface I mean a surface which is the union of finitely many faces. Each face is a closed subset of a plane of which the boundary is finitely many line segments. Two line segments intersect (if they intersect at all) in a vertex (a single point) and every line segment is contained in exactly two faces.

We assume that to every plane there is associated a normal direction $\vec{N}$ (so that the face is oriented). To every line segment in a face, we let $\vec{n}$ be the vector orthogonal to the line pointing into the face. Then we orient the line segment by $\vec{n} \times \vec{N}$. We assume that the line has the opposite orientation given by its inclusion in the other face to which it belongs.

Consider the following atlas. To every face, take the interior of the face and consider any isometry to the plane, which sends the normal $\vec{N}$ to the positive normal to the plane. To every interior point of a line segement, choose two half circles of the same radius centred at the point, contained in either face, and identify this with a circle of the same radius in the plane. Note that the transition functions are of the form $z \longrightarrow az + b$, which are holomorphic. For a vertex, let α_i be the internal angle at each incident line segment. Let α be the sum of the internal angles. Then we may identify the segments of the circles with a circle in the plane, using the map $z \longrightarrow z^{2\pi/\alpha}$. Note that since every vertex has only one chart, we don't need to check anything about the transition functions at the vertices themselves. On the other hand, this map is holomorphic away from zero.

We have already seen that is possible to attach a Riemann surface to a multi-valued locally holomorphic function. For a quite complicated example consider the pair of variables connected by

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right) \qquad \text{and} \qquad w = z - \sqrt{z^2 - 1}.$$

The square root is zero at $z = \pm 1$. We obtain a branch of w by cutting out the interval $[-1, 1]$. The Riemann surface associated to w is obtained by taking two copies of $\mathbb{C}$, cutting them along the interval $[-1, 1]$ and joining opposite edges. In fact the Riemann surface we get this way is a copy of $\mathbb{C}^*$, since it is just the w-plane. Note that something interesting happens above the two points ± 1. Above every other point there are two points but above these points there is just one point.

It is interesting to consider the Riemann surface associated to $\log z$. In this case we need infinitely many copies of $\mathbb{C}$, one for every integer multiple of $2\pi i$. These copies are joined by cutting along the positive real axis and joining the lower edge to the upper edge of the copy one level higher (so that we joining the upper edge to the lower edge of the copy one level lower). One again the resulting Riemann surface is not complicated, just a copy of $\mathbb{C}$. The map back is just

$$\mathbb{C} \longrightarrow \mathbb{C}^* \qquad \text{given by} \qquad w \longrightarrow e^w = z.$$

Suppose that we start with a polynomial in two variables $f(z, w)$. Provided the gradient doesn't vanish anywhere on the zero locus, the zero locus is a Riemann surface. Since the gradient doesn't vanish it follows that one of the two projections down to either the z-axis or the

w-axis has a continuous inverse, by the implicit function theorem. The transition functions are holomorphic, since solving for one variable in terms of the other, we get a holomorphic function.

For example, consider

$$f(z, w) = w^2 - 2wz + 1.$$

If we solve for w we get

$$w = z \pm \sqrt{z^2 - 1},$$

so this is just another way to think of the Riemann surface associated to the function

$$z - \sqrt{z^2 - 1}.$$

Given that there are so many Riemann surfaces it is somewhat surprising that:

Theorem 13.9 (Uniformisation Theorem)**.** *Let R be a simply connected Riemann surface.*

Then R is biholomorphic to one of

(1) The unit disk Δ.
(2) $\mathbb{C}$.
(3) The Riemann sphere $\mathbb{P}^1$.

Note that these three cases are mutually exclusive. (1) and (2) are not biholomorphic by Liouville's theorem. (3) is compact and (1) and (2) are not, so that (1) and (2) are not even homeomorphic to (3).

Consider how one might go about proving (13.9). Note that if we remove a point from $\mathbb{P}^1$ then we get $\mathbb{C}$. The key thing is then to prove that if R is a simply connected Riemann surface which is not compact then R is biholomorphic to an open subset of $\mathbb{C}$. In this case we can apply the Riemann mapping theorem to conclude we are in case (1) or (2).

An open subset of $\mathbb{C}$ has a global non-constant holomorphic function, namely the map $z \longrightarrow z$. On the other hand, if we have a global non-constant holomorphic function then we are clearly very close to showing that we have an open subset of $\mathbb{C}$. At the very least we get a holomorphic map $R \longrightarrow \mathbb{C}$. This is the important result to check, that every non-compact simply connected Riemann surface has a holomorphic function.

Now if we have a global holomorphic map we have a global harmonic map, just by taking the real part of the holomorphic map. Conversely on a simply connected Riemann surface a global harmonic map gives rise to a global holomorphic map. As usual the potential ambiguity between possible choices of a harmonic conjugate on different charts is not a problem on a simply connected Riemann surface.

So how to produce harmonic functions? Well harmonic functions correspond to potential functions in electrostatics. Imagine the following thought experiment; imagine that the Riemann surface is made out of metal. If you drop an electron onto the surface then the charge will distribute itself according to the usual laws of physics, so that you will get a potential. From the potential one can construct a harmonic function which goes to $-\infty$ at the point you put the electron. In a coordinate chart around the electron the harmonic function behaves like $\log r$.

So this gives us a clear line of attack. Prove that every simply connected Riemann surface admits a harmonic function which behaves like $\log r$ in a neighbourhood of one point. We will need to develop the theory of harmonic functions.