HOMEWORK #4, DUE WEDNESDAY FEBRUARY 25TH

1. Prove that on any region U the family of holomorphic functions $\mathfrak{F}$ with positive real part is normal. Under what added condition is it locally bounded? (*Hint: Consider the functions* e^{-f}.)
2. Show that the functions z^n, n a non-negative integer, form a normal family in $|z| < 1$, also in $|z| > 1$, but not in any region that contains a point of the unit circle.
3. If $f(z)$ is entire, show that the family

$$\mathfrak{F} = \{\, f(kz) \,|\, k \in \mathbb{C} \,\},$$

form a normal family in the annulus $r_1 < |z| < r_2$ if and only if f is a polynomial.
4. If the family $\mathfrak{F}$ of holomorphic functions is not normal on U then there is a point z_0 of U such that $\mathfrak{F}$ is not normal in a neighbourhood of z_0. (*Hint: A compactness argument.*)
5. If U is a simply connected region which is invariant under complex conjugation and z_0 is real prove that the map $f\colon U \longrightarrow \Delta$ given by the Riemann mapping theorem satisfies $f(\bar{z}) = \bar{f}(z)$.