HOMEWORK #2, DUE WEDNESDAY JANUARY 28TH

1. If u is harmonic and bounded in $0 < |z| < \rho$ then show that the origin is a removable singularity in the sense that u may be extended to a harmonic function on the whole disk $|z| < \rho$.

2. *(Hadamard's three circle theorem).* Suppose that $f(z)$ is analytic in the annulus

$$r_1 < |z| < r_2,$$

and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ on the circle $|z| = r$ then show that

$$M(r) \leq M(r_1)^{\alpha} M(r_2)^{1-\alpha} \qquad \text{where} \qquad \alpha = \frac{\log\left(\frac{r_2}{r}\right)}{\log\left(\frac{r_2}{r_1}\right)}.$$

Discuss cases of equality. (*Hint:* Apply the maximum principle to a linear combination of $\log|f(z)|$ and $\log|z|$.)

3. *(Poisson's integral for the half plane).* Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Prove that

$$P_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi)\, d\xi,$$

represents a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity.

4. Prove that any function which is harmonic and bounded on the upper half plane and continuous on the real axis, can be represented as a Poisson integral (*Remark:* some care is needed to deal with the behaviour at infinity, see Ahlfors page 171).