1. (15pts) State and prove the maximum principle for harmonic functions.

Solution: Let U be a region and let $u\colon U \longrightarrow \mathbb{R}$ be a harmonic function. If u achieves its maximum on U then u is constant.
Let $z_0 \in U$ be a point where u achieves its maximum. Since u is harmonic we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, \mathrm{d}\theta,$$

for any $r \geq 0$ such that the closed ball $|z - z_0| \leq r$ is contained in U. As u is continuous and the average value of $u(z)$ on the circle $|z - z_0| = r$ is $u(z_0)$ we must have $u(z) = u(z_0$ on the circle $|z - z_0| = r$.
Varying the radius r of the circle we must have that $u(z)$ is constant on the whole closed ball $|z - z_0| \leq r$. Varying z_0 we must have that $u(z)$ is constant.

2. (15pts) Show that

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty}(1 - p_n^{-s}).$$

where $\sigma = \operatorname{Re} s > 1$ and $p_1, p_2, \ldots$ are the primes in increasing order.

Solution: We first check absolute convergence of the product. This is equivalent to absolute convergence of the sum

$$\sum_{n=1}^{\infty} |p_n^{-s}| = \sum_{n=1}^{\infty} p_n^{-\sigma}.$$

If we compare this with

$$\sum_{n=1}^{\infty} n^{-\sigma}$$

we see that have absolute convergence for any $\sigma > \sigma_0 > 1$.
By definition of the Riemann zeta function we have

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $\sigma = \operatorname{Re} s > 1$. It follows that

$$\zeta(s)(1 - 2^{-s}) = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty}(2n)^{-s} = \sum_{m} m^{-s},$$

where m runs over the odd integers.
Similarly, by inclusion-exclusion,

$$\zeta(s)(1-2^{-s})(1-3^{-s}) = \sum_{n=1}^{\infty} n^{-s} - (2n)^{-s} - (3n)^{-s} + (6n)^{-s} = \sum_{m} m^{-s},$$

where m runs over the integers coprime to 2 and 3. More generally

$$\zeta(s)(1 - 2^{-s})(1 - 3^{-s}) \ldots (1 - p_N^{-s}) = \sum_{m} m^{-s} = 1 - p_{N+1}^{-s}(1 + \ldots),$$

where m runs over the integers coprime to the first N primes. Since the dots are the tails of a convergent sequence, letting N go infinity, we get

$$\zeta(s) \prod_{n=1}^{\infty}(1 - p_n^{-s}) = 1.$$

3. (10pts) Let U be a region. Exhibit a family of compact subsets which exhausts U.

Solution: Let

$$E_n = \{ z \in U \,|\, |z| \leq n \text{ and } |z - z_0| \geq 1/n \text{ for every } z_0 \notin U \}.$$

Then $E_n \subset E_{n+1}$ and E_n is bounded. Fix $z_0 \notin U$. Then

$$F = \{ z \in U \,|\, |z - z_0| \geq 1/n \},$$

is a closed subset. As the intersection of closed sets is closed, it follows that E_n is closed. But then E_n is compact as it is closed and bounded. Suppose that $z \in U$. Pick m such that $|z| < m$. Let $Z = \mathbb{C} - U$. Then Z is a closed subset and

$$Z' = \{ z \in Z \,|\, |z| < m + 1 \}$$

is closed and bounded, so that Z is compact. Suppose that Z' is non-empty. Then the infimum of the distance from Z' to z is achieved by $\delta > 0$. Any other point of Z has distance at least one from z_0. Pick $n > m$ and $1/n < delta$. Then $z \in E_n$ so that

$$\bigcup_{n \in \mathbb{N}} E_n = U.$$

Thus $E_1, E_2, \ldots$ exhaust U.

4. (15pts) Prove that a family $\mathcal{F}$ of locally bounded holomorphic functions on a region U is equicontinuous on any compact subset E.

Solution:
Let $z_0 \in E$. By assumption we may find $r > 0$ and $M > 0$ such that if $|z - z_0| < r$ then $z \in U$ and $|f(z)| \leq M$. Pick two points z and z' such that $|z - z_0| < r$ and $|z' - z_0| < r$. By Cauchy's integral formula

$$\begin{aligned} f(z) - f(z') &= \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z} \, \mathrm{d}w - \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z'} \, \mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)(z-z')}{(w-z)(w-z')} \, \mathrm{d}w \\ &= \frac{z-z'}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z)(w-z')} \, \mathrm{d}w. \end{aligned}$$

If we suppose that $|z - z_0| < r/2$ and $|z' - z_0| < r/2$ then we get

$$|f(z) - f(z')| \leq \frac{r}{2\pi} \left| \int_{|w-z_0|=r} \frac{f(w)}{(w-z)(w-z')} \, \mathrm{d}w \right| \leq 4M,$$

as $|w - z| > r/2$ and $|w - z'| > r/2$.

5. (20pts) (i) Let $U \neq \mathbb{C}$ be a simply connected region and let $z_0 \in U$. Construct an injective holomorphic function $f\colon U \longrightarrow \Delta$ such that $f(z_0) = 0$ and $f'(z_0) > 0$, where Δ is the unit disc.

Solution:
Pick $a \notin U$. As U is simply connected we can construct a holomorphic branch of the square root $\alpha(z) = \sqrt{z-a}$. Let $w \in \mathbb{C}$. If w belongs to the image of h then $-w$ does not. It follows that the image of α is contained in a half plane, $\operatorname{Im} e^{i\phi} w \geq 0$. Let $\beta(w) = e^{-i\phi} w$. Then the image of the composition $\beta \circ \alpha$ is contained in the upper half plane. Let $w_0 = \beta \circ \alpha(z_0)$. Let

$$\gamma(z) = \frac{z - w_0}{z - \bar{w}_0}.$$

Then γ is a Möbius transformation which sends the real axis to the unit circle and w_0 to 0. It follows that $\gamma \circ \beta \circ \alpha$ sends the upper half plane to the unit circle and sends z_0 to 0. α, β and γ are all injective so that the composition is injective and the derivative is nowhere zero. Suppose that the derivative of the composition at the origin is $re^{i\theta}$. Let

$$\delta(z) e^{-i\theta} z.$$

Then δ maps the unit disc to the unit disc, fixes the origin so that the composition $f = \delta \circ \gamma \circ \beta \circ \alpha$ is injective, sends U to the unit disc, $f(z_0) = 0$ and $f'(z_0) > 0$.

(ii) If the function f is not surjective then exhibit an injective holomorphic function $g\colon U \longrightarrow \Delta$ such that $g(z_0) = 0$ and $g'(z_0) > f'(z_0)$.

Solution:
Pick $a \in \Delta$ not belonging to the image of $f(z)$. As U is simply connected, we may find a holomorphic branch for

$$F(z) = \sqrt{\frac{f(z) - a}{1 - \bar{a} f(z)}}.$$

Note that F is the composition of f, the automorphism of the unit disc

$$z \longrightarrow \frac{z - a}{1 - \bar{a} z},$$

and the square root. Thus F is injective and $|F(z)| < 1$.
Let

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)}.$$

Note that G is the composition of F and the automorphism of the unit disc

$$z \longrightarrow \frac{z - F(z_0)}{1 - \overline{F(z_0)} z}.$$

Thus G is injective and $|G(z)| < 1$. Clearly $G(z_0) = 0$. Moreover $G'(z_0) > 0$. In fact

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |a|}{2\sqrt{|a|}} f'(z_0) > f'(z_0).$$

6. (20pts) (i) Let v_1 and v_2 be two subharmonic functions on a region U. Show that $\max(v_1, v_2)$ is subharmonic.

Solution: The maximum v of two continuous functions is continuous. Let u be a harmonic function. Suppose that $v - u$ has a maximum at z_0. We want to show that $v - u$ is constant. $v_i - u$ is subharmonic and so we may assume that $u = 0$. Suppose that $v(z_0) = v_1(z_0)$. If $z \in U$ then

$$v_1(z) \leq v(z) \leq v(z_0) = v_1(z_0).$$

Thus z_0 is a maximum of $v_1(z)$ and so $v_1(z)$ is constant. If $v(z) \neq v_1(z)$ in a neighbourhood of z_0 then we must have $v(z) = v(z_0)$ by continuity. But then $v_2(z)$ is constant and so v is constant.
(ii) Show that Dirichlet's problem does not always have a solution.

Solution: Let $U = \Delta^*$ be the punctured unit disc and let $f(\zeta)$ be equal to zero on the unit circle and 1 at the origin. Then U is a bounded region and f is a continuous function on the boundary. Let u be a harmonic function such that $\limsup_{z\to\zeta} u(z) = f(\zeta)$. Then u is bounded in a neighbourhood of the origin so that u extends to a harmonic function on the whole unit disc.
Since the values of the extended harmonic function are zero on the boundary of the disc it follows that $u \leq 0$ by the maximum principle. But then $\limsup_{z\to 0} u(z) \leq 0 < 1 = f(0)$. Then u is bounded in a neighbourhood of the

7. (10pts) Let $f_1, f_2, \ldots$ be the iterates of the function $\sin z$, so that $f_1(z) = \sin z$ and $f_{n+1}(z) = f_n(\sin z)$. Show that $f_1, f_2, \ldots$ is not locally bounded at the origin.

Solution: Let

$$g(z) = \sin(iz) = f_1(iz).$$

Then $g(z) = i \sinh(z)$. Thus

$$g_2(z) = f_2(iz) = \sin((i \sinh(z)) = \sinh(\sinh(z)).$$

Continuing in this way we see that if

$$g_n(z) = f_n(iz) \qquad \text{then} \qquad g_{n+1}(z) = g_n(\sinh(z)).$$

Let $h(x) = \sinh(x) - x$. Then

$$h'(x) = \cosh(x) - 1 \geq 0,$$

with equality if and only if $x = 0$. It follows that

$$\sinh(x) \geq x,$$

with equality if and only if $x = 0$. Pick any $x_0 > 0$. It follows that the sequence of numbers

$$x_n = g_n(x_0) \geq (1 + \delta)^n x_0,$$

for some fixed δ, depending on x_0. Thus $f_n(z)$ is not locally bounded.

Bonus Challenge Problems

8. (10pts) State and prove Ascoli-Arzola.

9. (10pts) Let U be a bounded region and let $f\colon \Gamma \longrightarrow \mathbb{R}$ be a function on the boundary which is bounded, $|f(\zeta)| \leq M$.
Show that the Perron function u associated to the Perron family $\mathcal{P}(f)$ is harmonic.