MODEL ANSWERS TO THE FIFTH HOMEWORK

1. We proved in class that if one blows up a point the log discrepancy is at least two.
Now let ν be an arbitrary valuation. Pick a birational morphism $\pi\colon T \longrightarrow S$, a composition of smooth blow ups, which realises the centre of ν as one of the exceptional divisors. Suppose that π is the composition of k blow ups, and that $E_1, E_2, \dots, E_k$ are the exceptional divisors, in the order in which they appear. We may assume that one cannot realise the centre of ν as a divisor with fewer than k blow ups. In this case the centre of ν is E_k and we may assume that $k > 1$.
Let $f\colon S_1 \longrightarrow S$ be the first blow up, and let $\pi_1\colon T \longrightarrow S_1$ be the induced birational morphism. Then π_1 is a composition of $k-1$ blow ups. By induction, we have

$$K_T + \sum_{i=2}^{k} E_i = \pi_1^* K_{S_1} + \sum b_i E_i,$$

where $b_i \geq 2$ with equality iff $i = 2$. On the other hand

$$K_{S_1} + F = f^* K_S + 2F,$$

where F is the exceptional divisor of the blow up f, so that the strict transform of F on T is E_1. Thus

$$E_1 + \sum_{i=2}^{k} c_i E_i = \pi_1^* F,$$

where c_i are positive integers. It follows that

$$\begin{aligned} K_T + \sum_{i=1}^{k} E_i &= K_T + E_1 + \sum_{i=2}^{k} E_i \\ &= \pi_1^*(K_{S_1} - F) + \sum (b_i + c_i) E_i. \end{aligned}$$

2. Pick m such that $m(K_X + \Delta)$ is Cartier. Then the log discrepancy takes values in the discrete set $\mathbb{Z}\langle 1/m \rangle$ and the result is clear.
It is also interesting to see what happens when $K_X + \Delta$ is not $\mathbb{Q}$-Cartier. Suppose that there is a valuation of irrational log discrepancy less than zero. I claim that the set of log discrepancies is then dense in the real numbers. As in the lectures, we may assume that $X = S$ is a smooth

surface and $\Delta = (1+\epsilon)C$, where C is a smooth curve, and $\epsilon > 0$ is irrational.
Blowing up along the repeated intersection of the exceptional divisor and the strict transform of C, we can create a component of coefficient $n\epsilon$, as in the lectures. In other words, we may assume that the coefficient of C is $n\epsilon$, for any $n > 0$. Now consider blowing up along the exceptional divisor, but away from C. After one blow up the coefficient is $n\epsilon - 1$. After m such blow ups the coefficient is $n\epsilon - m$. But the set

$$\{\, n\epsilon - m \,|\, (n,m) \in \mathbb{N}^2 \,\},$$

is dense in $\mathbb{R}$, for any positive irrational number ϵ.
3. (i) As C_2 is the union of the two axes, the log canonical threshold is one.
(ii) We have to write down a log resolution. Let $\pi\colon Y \longrightarrow X$ first blow up the singular point, then the intersection of the strict transform of C with the exceptional divisor and finally blow up the triple intersection of the strict transform of C, the strict transform of the old exceptional divisor and the new exceptional divisor. Label the exceptional divisor E_1, E_2 and E_3, in the order they appear. Then C has multiplicity 2, 3 and 6 along E_1, E_2 and E_3 respectively. On the other hand, the log discrepancy of E_1 is 2, or E_2 is 3 and of E_3 is 5. Thus

$$K_Y + \lambda D + E_1 + E_2 + E_3 = \pi^*(K_X + \lambda C) + (2-2\lambda)E_1 + (3-3\lambda)E_2 + (6-5\lambda)E_3.$$

So the largest value of λ, such that the log discrepancies $2(1-\lambda)$, $3(1-\lambda)$ and $(6-5\lambda)$ are all non-negative is $5/6$. The log canonical threshold is $5/6$.
(iii) In this case a log resolution is given by blowing up twice. The multiplicity of C along E_1 and E_2 is 2 and 4 and the log discrepancy is 2 and 3. Thus

$$K_Y + \lambda D + E_1 + E_2 = \pi^*(K_X + \lambda C) + (2-2\lambda)E_1 + (3-4\lambda)E_2.$$

The log canonical threshold is therefore $3/4$.
(iv) The log canonical threshold is $1/m + 1/n$. The easiest way to see this is to use weighted blowups (or toric geometry). In terms of toric geometry, suppose that we make the weighted blow up corresponding to inserting a vector of type (m,n). Suppose that the exceptional divisor is E. Then

$$K_Y + \lambda D + E = \pi^*(K_X + \lambda C) + aE,$$

where a is a function of λ. Now E is a copy of $\mathbb{P}^1$, but the twist is that Y is singular along E (in other words, the trick is to go to a log resolution, focus on the component which computes the log canonical threshold, and contract all the other components; the resulting surface

Y is then singular along E). Now there are then two singular points along E, and again by general theory, these singular points are cyclic quotient singularities of index m and n. This implies that

$$(K_Y + E)|_E = K_E + \sum \frac{m-1}{m} p + \frac{n-1}{n} q,$$

where p and q are the points of E corresponding to the two cyclic quotient singularities. On the other hand, D intersects E transversally in one point. Now at the log canonical threshold, $a = 0$ (indeed, $K_Y + E + \lambda D$ is log canonical). Thus

$$0 = (K_Y + E + \lambda D) \cdot E = -2 + \frac{m-1}{m} + \frac{n-1}{n} + \lambda.$$

Thus

$$\lambda = \frac{1}{m} + \frac{1}{n},$$

as conjectured.

4. Let $\pi \colon S \longrightarrow Z$ contract a real K_S-extremal ray. As in the classical case, there are three cases, given by the dimension of Z.

Suppose that Z is a point. Then S is either a copy of $\mathbb{P}^2$, defined over the reals, or a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, on which complex conjugation acts by switching the two fibrations.

Suppose that Z is a curve. Then Z is a curve over the reals. If p is a point of Z which is not equal to its complex conjugate, then the fibre over p is a copy of $\mathbb{P}^1$, with no real points. Otherwise if p is a real point, then there are two possibilities for the fibre. In the first, the fibre is a curve of genus zero over the reals (there are two such, those with real points $\mathbb{P}^1_{\mathbb{R}}$, and those with none, $x^2 + y^2 + z^2 = 0 \subset \mathbb{P}^2$). In the second the fibre is a union of two $\mathbb{P}^1$'s, joined at a real point, and complex conjugations switches the two copies. There is no limit to the number of these reducible fibres.

Finally suppose that Z is a surface. In this case, the exceptional locus is either irreducible, a curve of genus zero over the reals, or reducible, two complex conjugate -1-curves.