MODEL ANSWERS TO THE THIRD HOMEWORK

1. See question 3, hwk # 2.
2. (i) Pick $x \in X$. Then we may find $D' \in |D|$ and $H' \in |H|$ such that $x \notin D'$ and $x \notin H'$. But then $x \notin D' + H' \in |D + H|$. Thus $|D + H|$ is base point free.
Now suppose that x and $y \in X$, $x \neq y$. Pick $D' \in |D|$ and $H' \in |H|$ such that $y \notin D'$ and whilst $x \in H'$ but $y \notin H'$. Then $x \in D' + H' \in |D + H|$ whilst $y \notin D' + H'$, so that $\phi = \phi_{|D+H|}$ separates points.
Finally let z be an irreducible length two scheme, with support x. Pick $D' \in |D|$ and $H' \in |H|$ such that $x \notin D'$ and $x \in H'$ whilst $z \not\subset H$. Then $x \in D' + H' \in |D + H|$ whilst $z \not\subset D' + H'$. Thus ϕ separates tangent vectors.
Since ϕ separates points and tangent vectors, it follows that ϕ is an embedding.
(ii) Immediate from (i).
(iii) By (3.3.3) there is an integer m_0 such that $D + m_0H$ is semiample. But then

$$D + mH = D + m_0H + (m - m_0)H,$$

is ample for all $m > m_0$.
3. Let F be the codimension one support of the base locus of $|D|$, and let $M = D - F$. Then

$$|D| = |M| + F,$$

and the base locus of $|M|$ has codimension two or more. It is clear that $\phi_{|M|} = \phi_{|D|}$, so that if $|M|$ is free, $\phi_{|D|}$ extends to a morphism.
Now suppose that C is a curve. The map ϕ is given by some linear system. The mobile part is free, since C is a curve, and so ϕ extends to a morphism.
4. First a general observation. Elements of the linear system

$$|a\pi^*H - \sum_{i=1}^{d-1} m_iE_i|,$$

correspond to elements of the linear system

$$|aH|_Y|,$$

which have multiplicity m_i at p_i. But the map

$$|aH| \longrightarrow |aH|_Y|,$$

is surjective, since the obsruction to surjectivity is

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a-d)) = 0.$$

Putting all of this together, elements of the linear system

$$|a\pi^* H - \sum_{i=1}^{d-1} m_i E_i|,$$

correspond to elements of the linear system

$$|aH|,$$

which when restricted to Y have multiplicity m_i at p_i.
(iii) A hyperplane $H \subset \mathbb{P}^{n+1}$ that contains $p_1, p_2, \dots, p_{d-1}$ must contain the line l. It follows that that the base locus of $|mL|$ is p_d.
(i) clear from (iii).
(ii) Let $\tilde{m}$ be the strict transform of the line. Then

$$D \cdot \tilde{m} = H \cdot m - 1 = 0.$$

Thus D is not ample.
(iv) Now

$$K_Y = (K_{\mathbb{P}^{n+1}} + Y)|_Y = (d-n-2)H|_Y.$$

Thus

$$K_X = \pi^* K_Y + (n-1)\sum_{i=1}^{d-1} E_i = (d-n-2)\pi^* H|_Y + (n-1)\sum_{i=1}^{d-1} E_i.$$

It follows that

$$K_X + nD = (d-2)\pi^* H|_Y - \sum_{i=1}^{d-1} E_i.$$

Consider hypersurfaces $W \subset \mathbb{P}^{n+1}$ of degree $d-2$ containing the $d-1$ points $p_1, p_2, \dots, p_{d-1}$. If W does not contain l, then

$$W \cdot l \geq d-1,$$

a contradiction. Thus any hypersurface of degree $d-1$ containing the points $p_1, p_2, \dots, p_{d-1}$ must contain the line l. In particular it must contain the point p_d. Thus p_d is in the base locus of the linear system

$$|K_X + nD|.$$

5. It is proved in (3.8) that the second term is correct when X is a curve. Following the proof of (3.8), if we pick a very ample divisor

H such that $|D + H|$ is also very ample and we pick $H \in |H|$ and $G \in |D + H|$ general then by induction we have

$$\begin{aligned}\Delta P(m-1) &= \chi(G, \mathcal{O}_G(mD + E + H)) - \chi(H, \mathcal{O}_H(mD + E + H)) \\ &= \frac{D^n m^{n-1}}{(n-1)!} + \frac{D^{n-2} \cdot (G \cdot (K_X + G - 2(E + H)) - H \cdot (K_X + H - 2(E + H)))m^{n-2}}{2(n-2)!} \\ &= \frac{D^n m^{n-1}}{(n-1)!} + \frac{D^n m^{n-2}}{2(n-2)!} + \frac{D^{n-1} \cdot (K_X - 2E)m^{n-2}}{2(n-2)!} + \dots,\end{aligned}$$

so that

$$P(m) = \frac{D^n m^n}{n!} + \frac{D^{n-1} \cdot (K_X - 2E)m^{n-1}}{2(n-1)!} + \dots,$$

as required.