9. Birational invariants

Definition 9.1. *Let X be a normal projective variety and let D be a $\mathbb{Q}$-divisor. The* ***Iitaka dimension*** *of D is equal to the maximum dimension of the image of X under the linear systems $|mkD|$,*

$$\kappa(X, D) = \max_{m \in \mathbb{N}} \dim \phi_m(X),$$

where $\phi_m = \phi_{|mkD|} \colon X \longrightarrow \mathbb{P}^N$. Equivalently,

$$\kappa(X, D) = \limsup_{m \in \mathbb{N}} \frac{\log(h^0(X, \mathcal{O}_X(mkD)))}{\log m}.$$

The ***Kodaira dimension*** *of X is the Iitaka dimension of the canonical divisor, $\kappa(X) = \kappa(X, K_X)$.*

The ***plurigenera*** *of X are the dimensions of the space of global n-forms of weight m,*

$$P_m(X) = h^0(X, \mathcal{O}_X(mK_X)).$$

Just to be confusing $p_g(X) = P_1(X)$, the number of independent global n-forms.

The ***irregularity*** *of X is the dimension of the space of 1-forms,*

$$q(X) = h^0(X, \Omega^1_X).$$

Theorem 9.2. *Let $\phi \colon X \dashrightarrow Y$ be two smooth projective varieties which are birational.*

Then X and Y have the same invariants defined in (9.1).

Definition 9.3. *Let X be a variety. We say that X is*

- ***rational*** *if X is normal and birational to $\mathbb{P}^n$, some n.*
- ***unirational*** *if X is normal and there is a dominant rational map $\phi \colon \mathbb{P}^n \dashrightarrow X$, some n.*
- ***rationally connected*** *if for every two points x and y there is a morphism $f \colon \mathbb{P}^1 \longrightarrow X$ such that $f(0) = x$ and $f(\infty) = y$.*
- ***rationally chain connected*** *if for every two points x and y there are morphisms $f_i \colon \mathbb{P}^1 \longrightarrow X$, $0 \leq i \leq k$, such that $f_0(0) = x$ and $f_i(\infty) = f_{i+1}(0)$, $0 \leq i \leq k-1$ and $f_k(\infty) = y$.*
- ***uniruled*** *if for every point $x \in X$ there is a non-constant morphism $f \colon \mathbb{P}^1 \longrightarrow X$ such that $f(0) = x$.*

Remark 9.4. *It is clear that rational implies unirational, unirational implies rationally connected (indeed given x and $y \in \mathbb{P}^n$, the line connecting them shows that $\mathbb{P}^n$ is rationally connected and the image of a rationally connected variety is rationally connected), rationally connected implies rationally chain connected and rationally chain connected implies uniruled, unless X is a point.*

Note that X is uniruled if and only if there is a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$. If $X = \mathbb{P}^1 \times C$, where C is a curve of genus at least one, then X is uniruled but not rationally chain connected. If $X = \mathbb{P}^1 \cup \mathbb{P}^1$, where the two copies of $\mathbb{P}^1$ are joined at a point, then X is rationally chain connected but not rationally connected. More generally, if X is rationally connected then it is irreducible. Let S be the cone over an elliptic curve. Then S is rationally chain connected but not rationally connected. Let $\pi\colon T \longrightarrow S$ blow up the vertex. Then T is not rationally chain connected. On the other hand rationally connected is a birational invariant and to check that X is rationally connected it suffices to show that any two points belonging to an open subset can be connected by the image of $\mathbb{P}^1$. It is conjectured that rationally connected does not imply unirational. In fact every smooth quartic $X \subset \mathbb{P}^4$ is rationally connected but it is conjectured that a general smooth quartic is not unirational. This seems to be one of the hardest outstanding problems in birational geometry. Finally it is known that there are unirational varieties which are not rational (in fact every smooth cubic in $\mathbb{P}^4$ is unirational; the general one is not unirational. Further every smooth quartic in $\mathbb{P}^4$ is not rational and some of these are unirational. Many other examples are now known).

Lemma 9.5. *Let X be a smooth projective variety.*

If X is uniruled then X is covered by rational curves C, such that $K_X \cdot C < 0$.

In particular $\kappa(X) = -\infty$.

Proof. Let $\operatorname{Hom}(\mathbb{P}^1, X)$ denote the moduli space of morphisms from $\mathbb{P}^1$ to X. Then there is a natural evaluation map,

$$\operatorname{Hom}(\mathbb{P}^1, X) \times \mathbb{P}^1 \longrightarrow X,$$

which sends the pair (f, t) to the point $f(t) \in X$. By assumption there is a component B of $\operatorname{Hom}(\mathbb{P}^1, X)$, which parametrises non-constant morphisms, for which this map is dominant. In characteristic zero, the differential of the map $\alpha\colon B \times \mathbb{P}^1 \longrightarrow X$ is surjective at some point (f, t) of B. On the other hand, there is a natural action of $\operatorname{Aut}(\mathbb{P}^1)$ on $\operatorname{Hom}(\mathbb{P}^1, X)$ and on B. It follows that the differential of α is surjective on the whole orbit of $\operatorname{Aut}(\mathbb{P}^1)$. In particular the differential of α is surjective on the whole fibre $\mathbb{P}^1$ (since the action of $\operatorname{Aut}(\mathbb{P}^1)$ is transitive). But then there is an open subset U of B for which the differential of α is surjective on $U \times \mathbb{P}^1$. Cutting by hyperplanes, we may assume that α is finite, in which case the differential is then an isomorphism on the whole fibre over the open set U. Then

$$K_X \underset{f}{\cdot} C \leq K_C = K_{\mathbb{P}^1} = -2.$$

Suppose to the contrary that $B \in |mK_X|$. Pick $C \not\subset B$. Then

$$0 \leq B \cdot C = (mK_X) \cdot C < 0,$$

a contradiction. □

Remark 9.6. *If X is rationally connected then $q(X) = 0$, and in fact all symmetric powers of all p-forms vanish.*