5. Effective results

It is natural to ask for effective versions of very ampleness and vanishing. Such results exist if we work with particular divisors:

Theorem 5.1 (Kodaira vanishing)**.** *Let X be a smooth projective variety and let D be an ample divisor. Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0,$$

for $i > 0$.

Kodaira vanishing is one of the most fundamental and important results in higher dimensional geometry. To prove it one needs to use some analytic methods (some Hodge theory and positivity of various metrics) or some very deep results of Deligne and Illusie on Hodge theory in characteristic p. In fact Kodaira vanishing fails in general in characteristic p.

Using Kodaira vanishing one can prove a technically more powerful result, Kawamata-Viehweg vanishing, whose proof is purely algebraic and whose form dictates most of the definitions and shape of the whole subject of higher dimensional geometry.

Conjecture 5.2 (Fujita's conjecture)**.** *Let X be a smooth projective variety of dimension n and let D be an ample divisor.*

Then

(1) $K_X + (n+1)D$ is base point free.
(2) $K_X + (n+2)D$ is very ample.

It is easy to give examples which show that Fujita's conjecture is sharp:

Example 5.3. *Let $X = \mathbb{P}^n$ and $D = H$. Then*

$$K_X + dD \sim (d - n - 1)H.$$

Thus $K_X + dH$ is base point free if and only if $d \geq n+1$ and it is ample if and only if $d \geq n+2$.

Example 5.4. *Let C be a smooth curve and let $D = p$. Then $K_C + p$ is never base point free.*

Indeed I claim that p is a base point of $|K_C + p|$. To see this consider

$$|K_C| \longrightarrow |K_C + p| \qquad \text{given by} \qquad D \longrightarrow D + p.$$

This map is linear and injective. By Riemann-Roch

$$\dim |K_C| = (2g - 2 - g + 1) - 1 + h^1(C, \mathcal{O}_C(K_C)) = g - 2 + h^0(C, \mathcal{O}_C) = g - 1$$
$$\dim |K_C + p| = (2g - 1 - g + 1) - 1 + h^1(C, \mathcal{O}_C(K_C + p)) = g - 1 + h^0(C, \mathcal{O}_{C(-p)}) = g - 1.$$

Thus the map is surjective and

$$|K_C + p| = |K_C| + p.$$

But then p is a base point of the linear system $|K_C + p|$. On the other hand, similar calculations show that $|K_C+p+q|$ is a free linear system, but that

Example 5.5. *ϕ_{K_C+p+q} is not injective if $p \neq q$ and not an embedding if $p = q$, so that $K_C + p + q$ is never very ample.*

Here is a baby version of (5.2):

Lemma 5.6. *Let X be a smooth projective variety.*

If D is a very ample divisor then $K_X + (n+1)D$ is free, and $K_X + (n+2)D$ is very ample.

Proof. We first show (1). We proceed by induction on the dimension of X. Let $x \in X$ be a point of X and let $Y \in |D|$ be a general element containing $x \in X$. Then Y is smooth and

$$(K_X + (n+1)D)|_Y \sim (K_X + Y + nD)|_Y = K_Y + nD|_Y.$$

By induction $K_Y + nD|_Y$ is free. Thus we may find $D' \in |K_Y + nD|_Y|$ not containing x. There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X+nD) \longrightarrow \mathcal{O}_X(K_X+(n+1)D) \longrightarrow \mathcal{O}_Y(K_Y+nD) \longrightarrow 0.$$

The obstructions to surjectivity of the lifting map

$$H^0(X, \mathcal{O}_X(K_X + (n+1)D) \longrightarrow H^0(Y, \mathcal{O}_Y(K_Y + nD)),$$

live in

$$H^1(X, \mathcal{O}_X(K_X + nD)) = 0,$$

which vanishes by Kodaira vanishing. Thus D' lifts to an element D'' of $|K_X + (n+1)D|$ not containing x. Hence $|K_X + (n+1)D|$ is free.

It is not hard to check that (free) + (very ample) is very ample. Thus $K_X + (n+2)D$ is very ample. $\square$

(5.2) is known in dimension two (Reider's Theorem) and freeness is known in dimensions three (Ein-Lazarsfeld) and four (Kawamata). The best general results are due to Anghern and Siu:

Theorem 5.7. *Let X be a smooth projective variety and let D be an ample divisor.*

(1) $K_X + mD$ is free for $m > \binom{n+1}{2}$, and

(2) $K_X + mD$ separates points, for $m \geq \binom{n+2}{2}$.

Even to show that $K_X + 5D$ is very ample on a smooth threefold X seems very hard at the moment.