4. Asymptotic Riemann-Roch

Theorem 4.1 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety and let D and E be two integral Weil divisors.*

If D is Cartier then

$$P(m) = \chi(\mathcal{O}_X(mD + E)) = \frac{D^n m^n}{n!} + \frac{D^{n-1} \cdot (K_X - 2E) m^{n-1}}{2(n-1)!} \dots,$$

is a polynomial of degree at most $n = \dim X$, where dots indicate lower order terms.

Since the case of curves is a little bit special we treat this case separately:

Lemma 4.2. *Let C be a smooth curve of genus g and let D be a divisor of degree d.*

Then

$$\chi(\mathcal{O}_C(D)) = d - g + 1.$$

Proof. Let E be any divisor of degree e and let p be any point. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_C(-p) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Here $\mathcal{O}_p$ is a skyscraper sheaf, supported at the single point p. Twisting by the divisor $E + p$ we have

$$0 \longrightarrow \mathcal{O}_C(E) \longrightarrow \mathcal{O}_C(E + p) \longrightarrow \mathcal{O}_p(E) \longrightarrow 0.$$

Taking the long exact sequence associated to the short exact sequence and using the additivity of the Euler characteristic we have:

$$\chi(\mathcal{O}_C(E + p)) = \chi(\mathcal{O}_C(E)) + \chi(\mathcal{O}_p) = \chi(\mathcal{O}_C(E)) + 1,$$

where we used the fact that $h^1(C, \mathcal{O}_p) = 0$. Since the formula on the RHS of Riemann-Roch is linear it follows that the Riemann-Roch formula holds for E if and only if the Riemann-Roch formula holds for $E + p$.

Any divisor is the difference of two effective divisors $D = D_1 - D_2$, $D_i \geq 0$. If p is a point of the support of D_2 then it suffices to prove the formula for $D + p$. By induction on the degree of D_2 we reduce to the case $D = D_1 \geq 0$. If p is a point of the support of D it suffices to prove the result for $D - p$. By induction on the degree of D it suffices to prove the result when the degree is zero. But then $D = 0$ so that

$$\chi(\mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - g. \qquad \square$$

Lemma 4.3. *Let X be a normal variety and let H be a very ample divisor.*

If $Y \in |H|$ is general then Y is normal.

Proof. X is normal if and only if it is regular in codimension one and S_2. Y is smooth in codimension one by Bertini. As X is S_2 the set of points where X is not Cohen-Macaulay is of codimension three or more. As Y does not contain any of the generic points of this set, Y is S_2. □

Proof of (4.1). By induction on the dimension n of X. Suppose that $n = 1$. Then X is a smooth curve. Riemann-Roch for $mD + E$ then reads

$$\chi(\mathcal{O}_X(mD + E)) = md + e - g + 1 = am - b,$$

where

$$a = d = \frac{\deg D}{1!} \qquad \text{and} \qquad b = g - 1 - e = \frac{\deg(K_X - 2E)}{2 \cdot 1!}.$$

Now suppose that $n > 1$. Pick a very ample divisor H, which is a general element of the linear system $|H|$, such that $H + D$ is very ample and let $G \in |D + H|$ be a general element. Then G and H are normal projective varieties and there are two exact sequences

$$0 \longrightarrow \mathcal{O}_X(mD+E) \longrightarrow \mathcal{O}_X(mD+E+H) \longrightarrow \mathcal{O}_H(mD+E+H) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_X((m-1)D+E) \longrightarrow \mathcal{O}_X(mD+E+H) \longrightarrow \mathcal{O}_G(mD+E+H) \longrightarrow 0.$$

Hence

$$\begin{aligned} \chi(X, \mathcal{O}_X(mD + E)) - \chi(X, \mathcal{O}_X(mD + E + H)) &= -\chi(H, \mathcal{O}_H(mD + E + H)) \\ \chi(X, \mathcal{O}_X((m-1)D + E)) - \chi(X, \mathcal{O}_X(mD + E + H)) &= -\chi(G, \mathcal{O}_G(mD + E + H)), \end{aligned}$$

and taking the difference we get

$$\begin{aligned} P(m) - P(m-1) &= \chi(G, \mathcal{O}_G(mD + E + H)) - \chi(H, \mathcal{O}_H(mD + E + H)) \\ &= \frac{(D^{n-1} \cdot G - D^{n-1} \cdot H)m^{n-1}}{(n-1)!} + \dots \\ &= \frac{D^n m^{n-1}}{(n-1)!} + \dots, \end{aligned}$$

is a polynomial of degree $n-1$, by induction on the dimension. The result follows by standard results on the difference polynomial $\Delta P(m) = P(m+1) - P(m)$. □

It is fun to use similar arguments to prove special cases of Riemann-Roch.

Theorem 4.4 (Riemann-Roch)**.** *Let C be a smooth curve of genus g and let D be an integral divisor on X of degree d. Then*

$$h^0(C, \mathcal{O}_C(D)) = d - g + 1 + h^0(C, \mathcal{O}_C(K_C - D)).$$

Proof. Follows from Serre duality and (4.1). □

Theorem 4.5 (Riemann-Roch for surfaces)**.** *Let S be a smooth projective surface of irregularity q and geometric genus p_g over an algebraically closed field of characteristic zero. Let D be a divisor on S.*

$$\chi(S, \mathcal{O}_S(D)) = \frac{D^2}{2} - \frac{K_S \cdot D}{2} + 1 - q + p_g.$$

Proof. Pick a very ample divisor H such that $H + D$ is very ample. Let C and Σ be general elements of $|H|$ and $|H + D|$. Then C and Σ are smooth. There are two exact sequences

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D + H) \longrightarrow \mathcal{O}_C(D + H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D + H) \longrightarrow \mathcal{O}_\Sigma(D + H) \longrightarrow 0.$$

As the Euler characteristic is additive we have

$$\begin{aligned}
\chi(S, \mathcal{O}_S(D + H)) &= \chi(S, \mathcal{O}_S(D)) + \chi(C, \mathcal{O}_C(D + H)) \\
\chi(S, \mathcal{O}_S(D + H)) &= \chi(S, \mathcal{O}_S) + \chi(\Sigma, \mathcal{O}_\Sigma(D + H)).
\end{aligned}$$

Subtracting we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = \chi(\Sigma, \mathcal{O}_\Sigma(D + H)) - \chi(C, \mathcal{O}_C(D + H)).$$

Now

$$\begin{aligned}
\chi(\Sigma, \mathcal{O}_\Sigma(D + H)) &= (D + H) \cdot \Sigma - \deg K_\Sigma/2 \\
\chi(C, \mathcal{O}_C(D + H)) &= (D + H) \cdot C - \deg K_C/2,
\end{aligned}$$

applying Riemann-Roch for curves to both C and Σ. We have

$$(D + H) \cdot \Sigma = (D + H) \cdot H + (D + H) \cdot D,$$

and by adjunction

$$K_\Sigma = (K_S + \Sigma) \cdot \Sigma \qquad \text{and} \qquad K_C = (K_S + C) \cdot C.$$

So putting all of this together we get

$$\begin{aligned}
\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) &= (D + H) \cdot D + \frac{1}{2}((K_S + C) \cdot C - (K_S + \Sigma) \cdot \Sigma) \\
&= (D + H) \cdot D + \frac{1}{2} K_S \cdot (C - \Sigma) + \frac{1}{2}(H \cdot H - (H + D) \cdot (H + D)) \\
&= \frac{D \cdot D}{2} - \frac{1}{2} K_S \cdot D.
\end{aligned}$$

We have

$$c = \chi(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - q + p_g.$$

Here we used the highly non-trivial fact that

$$h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S) = q,$$

from Hodge theory and Serre duality

$$h^2(S, \mathcal{O}_S) = h^0(S, \omega_S) = p_g. \qquad \square$$

Remark 4.6. *One can turn Riemann-Roch for surfaces around and use the arguments in the proof of* (4.5) *to prove basic properties of the intersection number.*