16. Cone and Contraction Theorem

The cone and contraction theorem are valid for kawamata log terminal pairs. These results are due principally to Kawamata and Shokurov:

Definition 16.1. *Let $\pi\colon X \longrightarrow Z$ be a proper morphism and let D be an $\mathbb{R}$-Cartier divisor. We say that D is π-**big** if its restriction to the general fibre is big.*

*Let D be an $\mathbb{R}$-divisor. We say that D is π-**semiample** if there is a contraction $\psi\colon X \longrightarrow Y$ over Z such that $D = \psi^* H$, where H is an ample over Z, $\mathbb{R}$-divisor on Y.*

Note that if $\pi\colon X \longrightarrow Z$ is birational then every divisor is big over Z as the generic fibre is a point.

Theorem 16.2 (Kawamata-Viehweg vanishing)**.** *Let $\pi\colon X \longrightarrow Z$ be a projective morphism and let D be an integral $\mathbb{Q}$-Cartier divisor.*

If (X, Δ) kawamata log terminal, $D - (K_X + \Delta)$ is π-nef and π-big then $R^i\pi_*\mathcal{O}_X(D) = 0$ for $i > 0$.

Theorem 16.3 (Base point free theorem)**.** *Let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

If (X, Δ) kawamata log terminal, $K_X + \Delta$ is π-nef and Δ is π-big then $K_X + \Delta$ is π-semiample.

Corollary 16.4. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

If (X, Δ) is kawamata log terminal and $K_X + \Delta$ is π-nef and π-big then $K_X + \Delta$ is π-semiample.

We indicate how (16.4) is derived from (16.3). We will need a simple result about kawamata log terminal pairs:

Lemma 16.5. *Let (X, Δ) be a kawamata log terminal pair and let D be any $\mathbb{R}$-Cartier divisor.*

If $D \geq 0$ then we may find $\delta > 0$ such that $(X, \Delta + \delta D)$ is kawamata log terminal.

Proof. Pick a log resolution of $(X, \Delta + D)$, $\pi\colon Y \longrightarrow X$. By assumption if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta)$$

then $\lfloor \Gamma \rfloor \leq 0$. If $G = \pi^* D$ then

$$\pi^*(\delta D) = \delta \pi^* D = \delta G.$$

and so

$$K_Y + \Gamma + \delta G = \pi^*(K_X + \Delta + \delta D). \qquad \square$$

Proof of (16.4). By assumption $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. Pick $\delta > 0$ such that $(X, \Delta + \delta D)$ is kawamata log terminal. As $\Delta + \delta D$ is π-big we may apply (16.3) to

$$K_X + \Delta + \delta D \sim_{\mathbb{R}} (1 + \delta)(K_X + \Delta)$$

to conclude that $K_X + \Delta$ is π-semiample. $\square$

Theorem 16.6 (Cone Theorem)**.** *Let (X, Δ) be a kawamata log terminal pair and let $\pi \colon X \longrightarrow Z$ be a projective morphism.*

Then

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta \geq 0} + \sum_i R_i = \mathbb{R}^+[C_i],$$

where R_i are countably many extremal rays spanned by rational curves C_i contracted by π, such that $0 < -(K_X + \Delta) \cdot C_i \leq 2n$.

In particular if H is any π-ample divisor, then there are only finitely many of these curves such that $(K_X + \Delta + H) \cdot C_i < 0$.

We sketch a proof of a stronger version of (16.6). We will need some preliminary definitions and results:

Definition 16.7. *Let (X, Δ) be a log pair.*

*A **non kawamata log terminal place** is a valuation of log discrepancy at most zero. A **non kawamata log terminal centre** is the centre of a non kawamata log terminal place. We say that a non kawamata log terminal centre is **minimal** if it is minimal with respect to inclusion.*

*The **non kawamata log terminal locus** $\mathrm{Nklt}(X, \Delta)$ is the union of the non kawamata log terminal centres.*

In the case when (X, Δ) is log canonical we will also refer to a non kawamata log terminal place (respectively centre, respectively locus) as a log canonical place (respectively centre, respectively locus).

Example 16.8. *Let $(X = \mathbb{P}^2, \Delta = C)$ where C is a nodal cubic. Then (X, Δ) is log canonical and the non kawamata log terminal centres are C and the node. The node is minimal and the non kawamata log terminal locus is the C.*

We will need a basic result about the calculus of log canonical centres:

Theorem 16.9. *Let (X, Δ) be a log canonical pair.*

1. *There are only finitely many log canonical centres.*
2. *The intersection of two log canonical centres is a union of log canonical centres.*
3. *A minimal log canonical centre is normal.*

Theorem 16.10. *Let (X, Δ) be a log pair and let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

Then

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta\geq 0} + i_*\overline{\mathrm{NE}}(Z_{-\infty}) + \sum_i R_i = \mathbb{R}^+[C_i],$$

where $i\colon Z_\infty \longrightarrow X$ is the inclusion of the non kawamata log terminal locus and R_i are countably many extremal rays spanned by rational curves C_i contracted by π, such that $0 < -(K_X + \Delta) \cdot C_i \leq 2n$.

In particular if H is any π-ample divisor, then there are only finitely many of these curves such that $(K_X + \Delta + H) \cdot C_i < 0$.

Corollary 16.11. *Let (X, Δ) be a log pair and let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

If (X, Δ) is log canonical outside finitely many points then

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta\geq 0} + \sum_i R_i = \mathbb{R}^+[C_i],$$

where R_i are countably many extremal rays spanned by rational curves C_i contracted by π, such that $0 < -(K_X + \Delta) \cdot C_i \leq 2n$

In particular if H is any π-ample divisor, then there are only finitely many of these curves such that $(K_X + \Delta + H) \cdot C_i < 0$.

Proof. Immediate from (16.10), since $Z_{-\infty}$ contains no curves. □

The following key result is due to Kawamata:

Theorem 16.12. *Let (X, Δ) be a log pair where X is projective and kawamata log terminal. Let H be an ample divisor and let V be the normalisation of a non kawamata log terminal centre W.*

If (X, Δ) is log canonical at the generic point of W then we may write

$$(K_X + \Delta + H)|_V = K_V + \Theta,$$

where (V, Θ) is a log pair and the non kawamata log terminal locus of (V, Θ) is the restriction of the non kawamata log terminal locus of (X, Δ).

Definition 16.13. *Let (X, Δ) be a log canonical pair and let $D \geq 0$ be an $\mathbb{R}$-Cartier divisor. The* ***log canonical threshold*** *of (X, Δ) with respect to D is*

$$\lambda = \sup\{\, t \in \mathbb{R} \,|\, (X, \Delta + tD) \textit{ is log canonical}\}.$$

Proof of (16.10). We just prove the absolute case, that is, when Z is a point. As usual pick an ample divisor A such that if μ is the nef threshold of (X,Δ) with respect to A then $D = K_X + \Delta + \mu A = K_X + \Delta + H$ is zero on only one $(K_X+\Delta)$-extremal ray R.

Let $\nu = \nu(X,D)$ be the numerical dimension. There are two cases. If $\nu < n$, that is, if D is not big then we are looking for rational curves which cover X. We apply (14.7) to $D_1, D_2, \dots, D_n$,

$$D_i = \begin{cases} D & \text{if } i \leq \nu + 1 \\ H & \text{otherwise.} \end{cases}$$

With this choice, we have

$$D_1 \cdot D_2 \cdot \ldots D_n = 0$$

and

$$\begin{aligned} -K_X \cdot D_2 \cdot \ldots D_n &= -D_1 \cdot D_2 \cdot \ldots D_n + \Delta \cdot D_2 \cdot \ldots D_n + H \cdot D_2 \cdot \ldots D_n \\ &> 0. \end{aligned}$$

Thus (14.7) implies that X is covered by rational curves Σ such that

$$D \cdot \Sigma = 0 \qquad \text{and} \qquad H \cdot \Sigma \leq 2n \frac{H \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n}.$$

The first condition implies that Σ spans the extremal ray R. Using the first equality, we can rewrite the second inequality as

$$\begin{aligned} -(K_X + \Delta) \cdot \Sigma &= H \cdot \Sigma \\ &\leq 2n \frac{H \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n} \\ &= 2n \frac{-(K_X+\Delta) \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n} \\ &\leq 2n \frac{-K_X \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_n} \\ &= 2n. \end{aligned}$$

Now suppose that D is big. Pick G such that $H - G$ is ample, close enough to H such that G is ample and $K_X + \Delta + G$ is big. Then we may find $B \geq 0$ such that

$$B \sim_{\mathbb{R}} K_X + \Delta + G.$$

Consider the closed sets

$$Z_t = \operatorname{Nklt}(X, \Delta + G + tB).$$

If $t = 0$ then we get $Z_{-\infty}$ and if

$$t \leq s \qquad \text{then} \qquad Z_t \subset Z_s.$$

If t is large then Z_t is equal to the support of B and by Noetherian induction

$$\{ Z_t \,|\, t \in [0, \infty) \}$$

is a finite set. Let W be a closed irreducible subset with normalisation V and let $j\colon V \longrightarrow X$ be the composition of the normalisation and inclusion. We say that R *comes from* V if there is a ray S of $\overline{\mathrm{NE}}(V)$ such $i_*S = R$. In this case note that we can choose S extremal.

By construction $B \cdot R < 0$. It follows that $R = \mathbb{R}_{\geq 0}\alpha$ and $\beta \in \mathrm{NE}(X)$ is close enough to α then $B \cdot \beta < 0$ and we may write

$$\beta = \sum a_i [C_i] \qquad \text{where} \qquad B \cdot C_i < 0.$$

It follows that $C_i \subset B$ so that β comes from the normalisation V of a component W of B. But then R comes from the normalisation of a component V of B.

Pick V with the property that it is the normalisation of a component W of some Z_t, R comes from V and W is minimal with this property. If V is the normalisation of a component of $Z_0 = Z_{-\infty}$ then there is nothing to prove. Otherwise let λ be the log canonical threshold of $(X, \Delta + G)$ with respect to B at the generic point of V. By (16.12) we may find (V, Θ) such that

$$(K_X + \Delta + \lambda B + G)|_V = K_V + \Theta,$$

and

$$\mathrm{Nklt}(V, \Theta) = Z_{-\infty}|_V.$$

Clearly $(K_V + \Theta) \cdot S < 0$ and by assumption S does not come from $\mathrm{Nklt}(V, \Theta)$. Therefore we are done by induction on the dimension. $\square$