15. Kawamata log terminal and all that

Definition 15.1. *We say that a log pair* (X, Δ) *is* ***kawamata log terminal*** *if there is a log resolution* $\pi\colon Y \longrightarrow X$ *such that if we write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ *and* $E \geq 0$ *have no common components,* $\pi_*\Gamma = \Delta$*, and* $\pi_* E = 0$ *then* $\llcorner \Gamma \lrcorner = 0$*.*

We could rephrase this definition as saying that the coefficients of Δ lie between zero and one, and that this condition continues to hold on Y. If we rewrite the equation above as

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

note that the kawamata log terminal condition becomes $\lfloor \Gamma \rfloor \leq 0$.

In fact this condition holds on any birational model and we have:

Lemma 15.2. *A log pair* (X, Δ) *is kawamata log terminal if and only if the log discrepancy is greater than zero and* $\lfloor \Delta \rfloor = 0$*.*

Proof. Suppose that (X, Δ) is kawamata log terminal. We have to check that the log discrepancy of every valuation ν is greater than zero. If ν is exceptional for π then this is clear. Replacing (X, Δ) by (Y, Γ) it suffices to check that a log smooth pair (X, Δ) has log discrepancy greater than zero, if $\lfloor \Delta \rfloor = 0$. This follows from the formula for the log discrepancy of a blow up.

Now suppose that the log discrepancy is greater than zero. Let $\pi\colon Y \longrightarrow X$ be a log resolution. If we write

$$K_Y + \tilde{\Delta} + \sum E_i = \pi^*(K_X + \Delta) + \sum a_i E_i$$

then $a_i > 0$. Thus $\lfloor \Gamma \rfloor \leq 0$. □

Kawamata log terminal pairs behave very well with respect to finite morphisms:

Lemma 15.3. *Let* $\pi\colon Y \longrightarrow X$ *be a finite morphism and let* (X, Δ) *and* (Y, Γ) *be log pairs such that*

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

Then (X, Δ) *is kawamata log terminal if and only if* (Y, Γ) *is kawamata log terminal.*

Proof. The trick is to prove a much stronger result. Let us drop the condition that Δ and Γ are effective. We will then prove that (X, Δ) has log discrepancy at least zero if and only if (Y, Γ) has log discrepancy at least zero, with simultaneous equality.

We first prove that if (Y, Γ) is kawamata log terminal and π is Galois, with Galois group G, then (X, Δ) is kawamata log terminal. Let ν be a valuation of X. Pick a G-equivariant log resolution $g\colon V \longrightarrow Y$ of Y, which extracts the valuations $\mu_1, \mu_2, \dots, \mu_k$ corresonding to ν. Let $f\colon W \longrightarrow X$ be the quotient of g, so that there is a commutative square

$$\begin{array}{ccc} V & \xrightarrow{g} & Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{f} & X. \end{array}$$

Now f is not necessarily a log resolution. However it will extract ν and W is $\mathbb{Q}$-factorial. By assumption the log discrepancy of μ_i is at least zero. The Riemann-Hurwitz formula for log pairs then says that the log discrepancy of ν is at least zero, with equality if and only if we have equality for each μ_i.

Thus (X, Δ) is kawamata log terminal.

Now suppose that (X, Δ) is kawamata log terminal. Let $\psi\colon Z \longrightarrow X$ be the Galois closure of π. Then the induced morphism $Z \longrightarrow Y$ is Galois. Replacing Z by Y we may assume that π is Galois. The result is easy in this case.

Finally if (Y, Γ) is Galois then going up we may assume that π is Galois. □

Suppose that $\mathbb{Z}_r$ acts on $\mathbb{C}^n$. It turns out that we can always diagonalise the action:

$$(x_1, x_2, \dots, x_n) \longrightarrow (y_1, y_2, \dots, y_n),$$

where $y_i = \omega^{a_i} x_i$, and ω is a primitive rth root of unity. We can encode this by the datum:

$$\frac{1}{r}(a_1, a_2, \dots, a_n).$$

As usual, we can assume that $0 \leq a_i \leq r - 1$. Also, since we get to choose ω, if the action is faithful, then we can partialise normalise, and we may assume that $a_1 = 1$. Finally, we always assume that the action is unramified in codimension one, so that the gcd of all but one of the a_i, for any i, is always one. The number r is called the index of the quotient singularity.

For surfaces there are two interesting extreme cases:

$$\frac{1}{r}(1, r-1) \qquad \text{and} \qquad \frac{1}{r}(1, 1).$$

In the first case,

$$\mathbb{C}[x,y]^{\mathbb{Z}_r} = \mathbb{C}[x^r, y^r, xy] = \frac{\mathbb{C}[a,b,c]}{\langle ac = b^n \rangle}.$$

Using different coordinates, we have

$$(x^2 + y^2 - z^n = 0) \subset \mathbb{C}^3.$$

Suppose that we blow up the origin. We introduce coordinates s and t such that $x = sz$ and $y = tz$. Then we get

$$(s^2 + t^2 - z^{n-2}) \subset \mathbb{C}^3,$$

and an exceptional divisor, which is the union of two copies of $\mathbb{P}^1$, where the new singular point is the intersection of the two copies of $\mathbb{P}^1$. Continuing in this way, we get a chain of $n-1$, -2-curves. This is called an A_{n-1}-singularity.

We can encode the resolution by using a graph. The vertices are the exceptional divisors, and edges correspond to intersection of two exceptional divisors. We further label the vertices by minus the self-intersection of the exceptional divisors. An A_n-singularity corresponds therefore to a chain of n vertices, all labelled with 2.

At the opposite extreme, consider $1/r(1,1)$. Then

$$\mathbb{C}[x,y]^{\mathbb{Z}_r} = \mathbb{C}[x^r, x^{r-1}y, x^{r-2}y^2, \dots, y^r],$$

which is the coordinate ring of the cone over a rational normal curve of degree r. The minimal resolution consists of a single copy of $\mathbb{P}^1$, with self-intersection $-r$. The corresponding graph is a single vertex labelled by r.

Let S be any singular surface. The **minimal resolution** of S is a (the) relatively minimal model $\pi\colon T \longrightarrow S$ over S. That is, take any log resolution of S, and run a relatively minimal model program over S. The resulting morphism π is characterised by the property that it does not contract any -1-curves.

Theorem 15.4. *Let S be a cyclic quotient singularity of type $1/r(1,a)$.*

Then the graph of the minimal resolution of S is a chain of $\mathbb{P}^1$'s, labelled by $(a_1, a_2, \dots, a_k)$, where

$$\frac{r}{a} = a_1 - \frac{1}{a_2 - \dots},$$

a continued fraction.

For example, consider $1/11(1,5)$. We have

$$\begin{aligned}\frac{11}{5} &= 3 - \frac{4}{5}\\ &= 3 - \frac{1}{5/4}\\ &= 3 - \frac{1}{2 - 3/4}\\ &= 3 - \frac{1}{2 - \frac{1}{4/3}}\\ &= 3 - \frac{1}{2 - \frac{1}{2 - 2/3}}\\ &= 3 - \frac{1}{2 - \frac{1}{2 - \frac{1}{3/2}}}\\ &= 3 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2-1/2}}}\end{aligned}$$

Thus the minimal resolution is a chain of 5 $\mathbb{P}^1$'s, of self-intersection $(-3,-2,-2,-2,-2)$.

Theorem 15.5. *Let S be a surface, let C be a smooth curve on S, and suppose that S has cyclic quotient singularities of index $r_1, r_2, \dots, r_k$, such that the strict transform of C always intersects one end of the chain at a single point.*

Then

$$(K_S + C)|_C = K_C + \sum \frac{r_i - 1}{r_i} p_i,$$

where p_i are the points of C where S is singular, and the log discrepancy of the pair (S, C) is greater than zero (in fact equal to the minimum of $1/r_i$).

Proof. One can prove this in two ways. One is by direct computation, on the minimal resolution. The second is to use the Riemann-Hurwitz formula. □

Kawamata log terminal singularities are completely classified for surfaces.

Theorem 15.6. *Let S be a kawamata log terminal surface.*

Then the resolution graph of S is either a chain, or has one vertex of degree three, attached to three chains. If the indices of the chains *are p, q and r, then*

$$(p, q, r) = (2, 2, m), (2, 3, 3), (2, 3, 4), (2, 3, 5).$$

The log discrepancy is one if and only if each self-intersection is -2. The corresponding singularities are known as Du Val singularities, and the corresponding graphs are known as A_n, D_n, E_6, E_7 and E_8.

Proof. Suppose that there is a vertex of degree at least three. Let ν be the corresponding valuation. Then we can find a morphism $\pi\colon T \longrightarrow S$ which extracts precisely the exceptional divisor associated to ν (in other words, contract all other divisors on the minimal resolution). By assumption we may write

$$K_T + E = \pi^* K_S + aE,$$

where $a > 0$. It follows that $K_T + E$ is π-negative. Suppose that the singular points along E are cyclic quotient singularities $p_1, p_2, \dots, p_k$, with indices $r_1, r_2, \dots, r_k$. By adjunction, we have

$$0 > (K_T + E) \cdot E = K_E + \sum \frac{r_i - 1}{r_i} p_i = K_E + \Delta.$$

But then (E, Δ) is a log Fano pair, and the only possibilities have been listed.

With a little more work one can show that the only possibility is that each p_i is cyclic quotient, ie that otherwise $(S, (1 - a)E)$ is not kawamata log terminal. □

Corollary 15.7. *Let S be a normal surface.*

S is kawamata log terminal if and only if S has quotient singularities.

Proof. We already know that if S has quotient singularities then it is kawamata log terminal. Now suppose that S is kawamata log terminal. Then the resolution graph is given by (15.6). In characteristic zero the resolution graph determines the singularity and it is not hard to check that any graph in the list is a quotient singularity. □

Definition 15.8. *We say that a log pair (X, Δ) is* ***log canonical*** *if the log discrepancy is at least zero.*

Theorem 15.9. *Let S be a log canonical surface which is not kawamata log terminal.*

Then the minimal resolution of S is a smooth elliptic curve, a cycle of $\mathbb{P}^1$'s, a tree with a vertex of degree 3, and indices $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$, or two vertices of degree 3, connected by an interior chain, with two sets of -2-curves at the end, or a vertex with degree 4, attached to 4, -2-curves (the last case is really a degenerate case of the penultimate case).

Proof. In this case,

$$K_T + E = \pi^* K_S + \sum a_i E_i,$$

where $a_i \geq 0$ with equality at least once, and the result follows as in the proof of (15.6). □