13. MMP for surfaces

(12.3) allows us to define the K_S-MMP for surfaces. The aim of the minimal model program is to try to make K_S nef.

(1) Start with a smooth projective surface S.
(2) Is K_S nef? Is yes, then stop.
(3) Otherwise there is an extremal ray R of the cone of curves $\overline{\mathrm{NE}}(S)$ on which K_S is negative. By (12.3) there is a contraction $\pi\colon S \longrightarrow Z$ of R.
   - **Mori fibre space:** If $\dim Z \leq 1$ then the fibres of π are Fano varieties.
   - **Birational contraction:** In this case replace S by Z and return to (2).

In other words, the K_S-MMP produces a sequence of smooth surface $\pi_i\colon S_{i-1} \longrightarrow S_i$, where each π_i blows down a -1-curve (conversely each π_i blows up a smooth point of S_i), starting with $S_0 = S$. This process must terminate, since the Picard number of S_i is one less than the Picard number of S_{i-1}. At the end we have a smooth surface $T = S_k$, such that either K_T is nef or $\pi\colon T \longrightarrow C$ is $\mathbb{P}^1$-bundle over a curve, or $T \simeq \mathbb{P}^2$.

Definition 13.1. *Let X be a normal variety.*

We say that X is a ***Fano variety*** *if X is projective and $-K_X$ is ample.*

We say that a projective morphism $\pi\colon X \longrightarrow Z$ is a ***Fano fibration*** *if $-K_X$ is π-ample.*

*Let R be an extremal ray of the closed cone of curves of X. We say that R is K_X-**extremal** if $K_X \cdot R < 0$. We say $\pi\colon X \longrightarrow Z$ is the* ***contraction associated to*** *R if π is a contraction morphism and C is contracted if and only if $R = \mathbb{R}^+C$.*

Lemma 13.2. *Let S be a smooth surface. Then the log discrepancy of S is equal to 2 and the only valuations of log discrepancy 2 are given by blowing up a point.*

Proof. Easy calculation. □

Proof of (12.5). One direction is clear. If $\pi\colon S \longrightarrow T$ blows up p, and C is the exceptional divisor, then we have already seen that $C^2 = -1$, and $C \simeq \mathbb{P}^1$. But then by adjunction

$$2g - 2 = -2 = K_C = (K_S + C) \cdot C = K_S \cdot C - 1.$$

Thus $K_S \cdot C = -1$ and C is a -1-curve.

Now suppose that C is a -1-curve. Then $R = \mathbb{R}^+[C]$ is a K_S-extremal ray of the cone of curves. Let $\pi\colon S \longrightarrow T$ be the associated contraction morphism. Then T is smooth.

Now suppose that we write

$$K_S + C = \pi^* K_T + aC.$$

Dotting both sides by C, we see that $a = 2$, and we are done by (13.2). $\square$

The MMP for surfaces can be extended in two interesting (but essentially trivial) ways. The first way we restrict the choice of extremal rays to contract and the second way we group together extremal rays and contract them simultaneously.

First suppose we are given a projective morphism $g\colon S \longrightarrow U$. Then one can ensure that every step of the MMP lies over U, simply by only contracting rays of the relative cone of curves, $\overline{\mathrm{NE}}(S/U)$. At the end, either K_S is nef over U (meaning that it is nef on every curve contracted over U) or we get a Mori fibre space over U.

Secondly suppose we have a group G acting on S. By simultaneously contracting whole faces of the cone of curves, which are orbits of a single extremal ray, the resulting contraction is then G-equivariant. This gives us a K_S-MMP which preserves the action of G. Note though, that the relative Picard number of each step can be larger than one (in fact the relative Picard number of the G-invariant part is always one).

A particularly interesting case, is when S is a smooth surface defined over the real numbers. In this case, we let G be the Galois group of $\mathbb{C}$ over $\mathbb{R}$ (namely $\mathbb{Z}_2$, generated by complex conjugation). The resulting steps of the MMP respect the action of complex conjugation, so that the MMP is defined over $\mathbb{R}$. Clearly similar remarks hold for other non-algebraically closed fields.

Theorem 13.3 (Hodge Index Theorem)**.** *Let S be a smooth projective surface.*

Then the intersection pairing

$$\mathrm{NS}(S) \times \mathrm{NS}(S) \longrightarrow \mathbb{R},$$

has signature $(+, -, -, -, \dots, -)$.

In particular if $D^2 > 0$ and $D \cdot E = 0$ then $E^2 \leq 0$ with equality if and only if E is numerically trivial.

Proof. It suffices to prove the last statement for any D such that $D^2 > 0$. So we may assume that D is ample. Suppose that $E^2 \geq 0$.

Suppose that $E^2 > 0$. Consider $H = D + mE$, where m is large. As $H \cdot E > 0$, it follows that $\kappa(S, E) > 0$, by Asymptotic Riemann-Roch. But then $D \cdot E > 0$, a contradiction.

Now suppose that $E^2 = 0$ but that E is not numerically trivial. Then there is a curve C such that $E \cdot C \neq 0$. Let $C' = (D \cdot C)D - (D^2)C$. Then $C' \cdot D = 0$ and $E \cdot C' \neq 0$. Replacing C by C' we may assume that $D \cdot C' = 0$.

Let $E' = mE + C$. Then $D' \cdot E' = 0$ and

$$E'^2 = 2mE \cdot C + C^2.$$

Since $D \cdot E \neq 0$ we can choose m so that $E'^2 \neq 0$. Thus replacing E by E' we are reduced to the case when $E^2 > 0$. $\square$

Lemma 13.4 (Negativity of Contraction)**.** *Let $\pi\colon X \longrightarrow U$ be a proper birational morphism of varieties and let B be an $\mathbb{R}$-Cartier divisor.*

If $-B$ is π-nef then $B \geq 0$ if and only if $\pi_* B \geq 0$.

Proof. One direction is clear, if $B \geq 0$ then $\pi_* B \geq 0$.

Otherwise, we may assume that X and U are normal, and U is affine. Cutting by hyperplanes, we may assume that U is a surface. Passing to a resolution of X, we may assume that X is a smooth surface. Compactifying X and U we may assume that X and U are projective. Let $D = \pi^* H$ and list the exceptional divisors $E_1, E_2, \dots, E_k$. Then $D^2 > 0$ and $D \cdot E_i = 0$. It follows that the intersection matrix $(E_i \cdot E_j)$ is negative definite. Suppose that $B = \sum b_i E_i + B'$, where no component of $B' \geq 0$ is exceptional. Then

$$(\sum b_i E_i) \cdot E_j \leq B \cdot E_j < 0.$$

Thus $b_i \geq 0$. $\square$

Theorem 13.5 (Strong Factorisation)**.** *Let $\phi\colon S \dashrightarrow S'$ be a birational map between two smooth projective surfaces.*

Then there are two birational maps $p\colon T \longrightarrow S$ and $q\colon T \longrightarrow S'$ which are both compositions of smooth blow ups of smooth points (and isomorphisms) and a commutative diagram

$$\begin{array}{ccc} & T & \\ {}^{p}\swarrow & & \searrow^{q} \\ S & \overset{\phi}{\dashrightarrow} & S'. \end{array}$$

Proof. By elimination of indeterminancy we may assume that q is a composition of smooth blow ups. Replacing ϕ by p, we may therefore assume that $\phi\colon T \longrightarrow S$ is a birational morphism.

Consider running the K_W-MMP over S'. This terminates with a relative minimal model, $\pi\colon W \longrightarrow T$ over S'. The morphism π contracts -1-curves, and so π is a composition of smooth blow ups. It suffices to show that $T = S'$; we will only use the fact that K_T is nef over S'.

Suppose not. We may write

$$K_T + E = \pi^* K_{S'} + \sum a_i E_i.$$

Since S' is smooth it has log discrepancy two and so each $a_i \geq 2$. But then if we write

$$K_T = \pi^* K_{S'} + F,$$

then

$$F = \sum (a_i - 1) E_i \geq 0,$$

contains the full exceptional locus. By negativity of contraction, there is an exceptional divisor E_i such that $B \cdot E_i < 0$. But then K_T is not nef, a contradiction. □

Lemma 13.6. *Let (X, Δ) be a log pair.*

If X is a curve and $K_X + \Delta$ is nef then it is semiample.

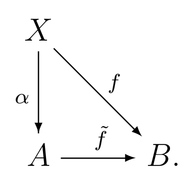
Proof. Let ν be the numerical dimension of $K_X + \Delta$, and let d be the degree. There are two cases:

(1) $d = 0$ and $\nu = 0$.
(2) $d > 0$ and $\nu = 1$.

If $d > 0$ then $K_X + \Delta$ is ample and there is nothing to prove. If $d = 0$ there are two cases. If $g = 1$ then Δ is empty and $K_X \sim 0$ as X is an elliptic curve. If $g = 0$ then $X \simeq \mathbb{P}^1$. Pick $m > 0$ such that $D = m(K_X + \Delta)$ is integral. Then $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1}$, so that $|D|$ is base point free. □

Definition 13.7. *Let X be a smooth projective variety. Then there is a morphism $\alpha\colon X \longrightarrow A$ to an abelian variety, which is universal amongst all such morphisms in the following sense:*

Let $f\colon X \longrightarrow B$ be another morphism to an abelian variety. Then there is a morphism $\tilde{f}\colon A \longrightarrow B$ and a commutative diagram



In characteristic zero, α induces an isomorphism

$$H^1(X, \mathcal{O}_X) \simeq H^1(A, \mathcal{O}_A).$$

In particular $\dim A = q(X)$.

Lemma 13.8 (Kodaira's Formula)**.** *Let $\pi\colon S \longrightarrow C$ be a contraction morphism, where S is a smooth projective surface and C is a smooth projective curve and the generic fibre is an elliptic curve.*

If K_S is nef over C then there is a divisor $\Delta \geq 0$ on C such that

$$K_S = \pi^*(K_C + \Delta).$$

Sketch of proof of (12.6). Let $\nu = \nu(S, K_S)$ be the numerical dimension. There are three cases.

If $\nu = 0$ then K_S is numerically trivial. Let $\alpha\colon S \longrightarrow A$ be the Albanese morphism. Let Z be the image. There are three cases, given by the dimension of Z.

If $q = 0$, equivalently Z is a point, then every numerically trivial divisor is torsion and there is nothing to prove. Suppose that $Z = C$ is a curve. Let F be a general fibre. Then

$$2g - 2 = K_F = (K_S + F) \cdot F = 0,$$

so that $g = 1$ and F is an elliptic curve. But then the result follows by (13.8) and (13.6). Finally suppose that Z is a surface. With some work, one shows that $Z = A$, and that α is birational, whence an isomorphism.

If $\nu = 1$ we first assume that $q = 0$. But then,

$$\chi(S, \mathcal{O}_S) = 1 - q + p_g > 0.$$

Riemman-Roch then reads

$$h^0(S, \mathcal{O}_S(mK_S)) \geq \chi(S, \mathcal{O}_S(mK_S)) = \chi(S, \mathcal{O}_S)) > 0.$$

It follows that $|mK_S| \neq \varnothing$, for $m > 0$. Let $C \in |mK_S|$. Then

$$2g - 2 = K_C = (K_S + C) \cdot C = (m+1)C|_C = 0.$$

But then C is a smooth curve of genus one (or a rational curve with a single node or cusp). Moreover since $K_C \sim 0$ it follows that $C|_C$ is torsion. Now there is an exact sequence,

$$0 \longrightarrow \mathcal{O}_S((k-1)C) \longrightarrow \mathcal{O}_S(kC) \longrightarrow \mathcal{O}_C(kC) \longrightarrow 0.$$

Note that $\mathcal{O}_C(kC) = \mathcal{O}_C$ infinitely often. Therefore $h^1(C, \mathcal{O}_C(kC)) \neq 0$ infinitely often. It follows that $h^1(S, \mathcal{O}_S(kC))$ is an unbounded function of k. Since

$$\chi(S, \mathcal{O}_S(kC)) \geq 0 \qquad \text{and} \qquad h^2(S, \mathcal{O}_S(kC)) = 0,$$

for $k \geq 2$, it follows that $h^0(S, \mathcal{O}_S(kC))$ is an unbounded function of k and we are done by (12.10). If $\nu = 1$ and $q > 0$, then we again have to carefully analyse the map α.

If $\nu = 2$ then $K_S^2 > 0$. If K_S is ample there is nothing to prove. Otherwise, by Nakai-Moishezon, there is a curve C such that $K_S \cdot C = 0$.

By Kodaira's Lemma, $K_S \sim A+E$, where A is ample and E is effective. But then C is a component of E and C has negative self-intersection. We have

$$2g - 2 = K_C = (K_S + C) \cdot C < 0.$$

But then $g = 0$, $C \simeq \mathbb{P}^1$, and $C^2 = -2$ (C is then called a -2-curve).

With some work, we can contract C, as before, $\pi\colon S \longrightarrow T$. In fact $K_S = \pi^* K_T$. Repeating this process, as in the K_S-MMP, we reduce to the case when K_S is ample. The only twist is that if we contract any curves, the resulting surface is necessarily singular. □

Proof of (12.7). Modulo some interesting details which we will skip, this essentially follows by applying the K_S-MMP and considering the maps given by abundance and the Albanese. □